THE CONTINUUM LIMIT OF A FERMION SYSTEM INVOLVING NEUTRINOS: WEAK AND GRAVITATIONAL INTERACTIONS

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Abstract. We analyze the causal action principle for a system of relativistic fermions composed of massive Dirac particles and neutrinos. In the continuum limit, we obtain an effective interaction described by a left-handed, massive SU(2)-gauge field and a gravitational field. The off-diagonal gauge potentials involve a unitary mixing matrix, which is similar to the Maki-Nakagawa-Sakata matrix in the standard model.

Contents

1. Introduction

In [4] it was proposed to formulate physics based on a new action principle in space-time. In the paper [7], this action principle was worked out in detail in the so-called continuum limit for a simple model involving several generations of massive Dirac particles. In the present article, we extend this analysis to a model which includes neutrinos. In the continuum limit, we obtain an effective interaction described by a left-handed massive SU(2) gauge field and a gravitational field.

More specifically, we again consider the causal action principle introduced in [4]. Thus we define the causal Lagrangian by

$$\mathcal{L}[A_{xy}] = |A_{xy}^2| - \frac{1}{8}\,|A_{xy}|^2\,, \tag{1.1}$$

where $A_{xy} = P(x,y)\,P(y,x)$ denotes the closed chain corresponding to the fermionic projector $P(x,y)$, and $|A| = \sum_{i=1}^8 |\lambda_i|$ is the spectral weight (where λ_i are the eigenvalues of A counted with algebraic multiplicities). We introduce the action $\mathcal{S}$ and the constraint $\mathcal{T}$ by

$$\mathcal{S}[P] \;=\; \iint_{M\times M} \mathcal{L}[A_{xy}]\, d^4x\, d^4y\,, \qquad \mathcal{T}[P] \;=\; \iint_{M\times M} |A_{xy}|^2\, d^4x\, d^4y\,, \tag{1.2}$$

where $(M, \langle .,. \rangle)$ denotes Minkowski space. The causal action principle is to

$$\text{minimize } \mathcal{S} \text{ for fixed } \mathcal{T}\,. \tag{1.3}$$

This action principle is given a rigorous meaning in [7, Section 2]. Every minimizer is a critical point of the so-called auxiliary action

$$\mathcal{S}_\mu[P] = \iint_{M\times M} \mathcal{L}_\mu[A_{xy}]\, d^4x\, d^4y\,, \qquad \mathcal{L}_\mu[A_{xy}] = |A_{xy}^2| - \mu\,|A_{xy}|^2\,, \tag{1.4}$$

which involves a Lagrange multiplier $\mu \in \mathbb{R}$.

We model the configuration of the fermions by a system consisting of a doublet of two sectors, each composed of three generations. Thus we describe the vacuum by the fermionic projector

$$P(x,y) = P^N(x,y) \oplus P^C(x,y)\,, \tag{1.5}$$

where the *charged sector* P^C is formed exactly as the fermionic projector in [7] as a sum of Dirac seas, i.e.

$$P^C(x,y) = \sum_{\beta=1}^{3} P_{m_\beta}(x,y)\,, \tag{1.6}$$

where m_β are the masses of the fermions and P_m is the distribution

$$P_m(x,y) = \int \frac{d^4k}{(2\pi)^4}\, (\slashed{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\,. \tag{1.7}$$

For the *neutrino sector* P^N we consider two different ansätze. The first ansatz of *chiral neutrinos* is to take a sum of left-handed, massless Dirac seas,

$$P^N(x,y) = \sum_{\beta=1}^{3} \chi_L\, P_0(x,y)\,. \tag{1.8}$$

The configuration of Dirac seas (1.5), (1.6) and (1.8) models precisely the leptons in the standard model. It was considered earlier in [4, §5.1]. The chiral ansatz (1.8) has the shortcoming that the neutrinos are necessarily massless, in contradiction to experimental observations. In order to describe massive neutrinos, we proceed as follows. As the mass mixes the left- and right-handed spinor components in the Dirac equation, for massive Dirac particles it is impossible to to restrict attention to one chirality. This leads us to the ansatz of *massive neutrinos*

$$P^N(x,y) = \sum_{\beta=1}^{3} P_{\tilde{m}_\beta}(x,y)\,. \tag{1.9}$$

Here the neutrino masses $\tilde{m}_\beta \geq 0$ will in general be different from the masses m_β in the charged sector. Except for the different masses, the ansätze (1.6) and (1.9) are exactly the same. In particular, it might seem surprising that (1.9) does not distinguish the left- or right-handed component, in contrast to the observation that neutrinos are always left-handed. In order to obtain consistency with experiments, if working with (1.9) we need to make sure that the *interaction* distinguishes one chirality. For example, if we described massive neutrinos by (1.9) and found that the neutrinos only couple to left-handed gauge fields, then the right-handed neutrinos, although being present in (1.9), would not be observable. With this in mind, working with (1.9) seems a possible approach, provided that we find a way to break the chiral symmetry in the interaction. It is a major goal of this paper to work out how this can be accomplished.

Working out the continuum limit for the above systems gives the following results. First, we rule out the chiral ansatz (1.8) by showing that it does not admit a global minimizer of the causal action principle. Thus in the fermionic projector approach, we must necessarily work with the massive ansatz (1.9). We find that at least one of the neutrino masses $\tilde{m}_\beta$ must be strictly positive. In order to break the chiral symmetry, we introduce additional right-handed states into the neutrino sector. It is a delicate question how this should be done. We discuss different approaches, in particular the so-called shear states and general surface states. The conclusion is that if the right-handed states and the regularization are introduced suitably, then the continuum limit is well-defined. Moreover, the structure of the effective interaction in the continuum

limit is described as follows. The fermions satisfy the Dirac equation coupled to a left-handed SU(2)-gauge potential A_L,

$$\left[i\partial\!\!\!/ + \chi_R \begin{pmatrix} A\!\!\!/_L^{11} & (A\!\!\!/_L^{21}\, U_{\text{MNS}})^* \\ A\!\!\!/_L^{21}\, U_{\text{MNS}} & -A\!\!\!/_L^{11} \end{pmatrix} - mY \right] \Psi = 0 \,, \tag{1.10}$$

where we used a block matrix notation (where the matrix entries are 3×3-matrices). Here mY is a diagonal matrix composed of the fermion masses,

$$mY = \text{diag}(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3,\; m_1, m_2, m_3) \,,$$

and U_{MNS} is a unitary 3×3-matrix. In analogy to the standard model, we refer to U_{MNS} as the Maki-Nakagawa-Sakata (MNS) matrix. The gauge potentials A_L satisfy a classical Yang-Mills equation coupled to the fermions. More precisely, writing the isospin dependence of the gauge potentials according to $A_L = \sum_{\alpha=1}^3 A_L^\alpha \sigma^\alpha$ in terms of Pauli matrices, we obtain the field equations

$$\partial^k{}_l (A_L^\alpha)^l - \Box (A_L^\alpha)^k - M_\alpha^2\, (A_L^\alpha)^k = c_\alpha\, \overline{\Psi} \big(\chi_L \gamma^k\, \sigma^\alpha \big) \Psi \,, \tag{1.11}$$

valid for $\alpha = 1, 2, 3$. Here M_α are the bosonic masses and c_α the corresponding coupling constants. The masses and coupling constants of the two off-diagonal components are equal, i.e. $M_1 = M_2$ and $c_1 = c_2$, but they may be different from the mass and coupling constant of the diagonal component $\alpha = 3$.

Moreover, our model involves a gravitational field described by the Einstein equations

$$R_{jk} - \frac{1}{2}\, \mathrm{R}\, g_{jk} + \Lambda\, g_{jk} = \kappa\, T_{jk} \,, \tag{1.12}$$

where R_{jk} denotes the Ricci tensor, R is scalar curvature, and T_{jk} is the energy-momentum tensor of the Dirac field. Moreover, κ and Λ denote the gravitational and the cosmological constants, respectively. We find that the gravitational constant scales like $\kappa \sim \delta^{-2}$, where δ is the length scale on which the shear and general surface states become relevant. The dynamics in the continuum limit is described by the coupled Dirac-Yang/Mills-Einstein equations (1.10), (1.11) and (1.12). These equations are of variational form, meaning that they can be recovered as Euler-Lagrange equations corresponding to an "effective action." The effective continuum theory is manifestly covariant under general coordinate transformations.

For ease in notation, the field equations (1.11) (and similarly the Einstein equations (1.12)) were written only for one fermionic wave function Ψ. But clearly, the equations hold similarly for many-fermion systems (see Theorem 7.1). In this context, it is worth noting that, although the states of the Dirac sea are explicitly taken into account in our analysis, they do not enter the Einstein equations. Thus the naive "infinite negative energy density" of the sea drops out of the field equations, making it unnecessary to subtract any counter terms.

Similar as explained in [7] for an axial field, we again obtain corrections to the field equations which are nonlocal and violate causality in the sense that the future may influence the past. Moreover, for a given regularization one can compute the coupling constant, the bosonic mass, and the gravitational constant.

We note that in this paper, we restrict attention to explaining our computations and results; for all conceptual issues and more references we refer to [7] and the survey article [9].

2. Regularizing the Neutrino Sector

In this section, we explain how the neutrino sector is to be regularized. We begin in §2.1 by reviewing the regularization method used in [4]. Then we give an argument why this method is not sufficient for our purposes (see §2.2). This leads us to extending our methods (see §2.3), and we will explain why these methods only work for the ansatz of massive neutrinos (see §2.4). In §2.5 we introduce the resulting general regularization scheme for the vacuum neutrino sector. In §2.6 we explain how to introduce an interaction, relying for the more technical aspects on Appendix A. Finally, in §2.7 we introduce a modification of the formalism of the continuum limit which makes some computations more transparent.

2.1. A Naive Regularization of the Neutrino Sector.

As in [7, Section 3] we denote the regularized fermionic projector of the vacuum by P^ε, where the parameter ε is the length scale of the regularization. This regularization length can be thought of as the Planck length, but it could be even smaller. Here we shall always assume that P^ε is *homogeneous*, meaning that it depends only on the difference vector $\xi := y-x$. This is a natural physical assumption as the vacuum state should not distinguish a specific point in space-time. The simplest regularization method for the vacuum neutrino sector is to replace the above distribution $P^N(x,y)$ (see (1.8)) by a function P^N_ε which is again left-handed,

$$P^N_\varepsilon(x,y) = \chi_L \, g_j(\xi) \, \gamma^j \; . \tag{2.1}$$

Such a regularization, in what follows referred to as a *naive regularization*, was used in [4] (see [4, eq. (5.3.1)]). It has the effect that the corresponding closed chain vanishes due to so-called *chiral cancellations* (see [4, eq. 5.3.2]),

$$A^N_{xy} := P^N_\varepsilon(x,y)\, P^N_\varepsilon(y,x) = \chi_L \,\not{g}(x,y)\, \chi_L \,\not{g}(y,x) \;=\; \chi_L\, \chi_R \,\not{g}(x,y)\, \not{g}(y,x) \;=\; 0 \,.$$

Regularizing the charged sector as explained in [4, Chapter 4] or [7], the closed chain of the regularized fermionic projector P^ε of the whole system is of the form

$$A_{xy} = P^\varepsilon(x,y)\, P^\varepsilon(y,x) = 0 \oplus A^C_{xy} \,.$$

Hence the closed chain has the eigenvalue zero with multiplicity four as well as the non-trivial eigenvalues λ_+ and λ_-, both with multiplicity two (see [4, §5.3]). Let us recall from [4, Chapter 5] how by a specific choice of the Lagrange multiplier μ we can arrange that the EL equations are satisfied: The operator Q corresponding to the action (1.4) is computed by (see [4, §3.5] or [7, Section 6])

$$Q(x,y) = (-4\mu) \oplus \Big[(1-4\mu)\Big] \sum_{s=\pm} \overline{\lambda_s} F_s \; P(x,y).$$

In order for the operator Q to vanish on the charged sector, we must choose

$$\mu = \frac{1}{4} \,. \tag{2.2}$$

Then

$$Q(x,y) = -\sum_{s=\pm} \overline{\lambda_s} F_s \; P^N_\varepsilon(x,y) \;\oplus\; 0,$$

and multiplying by $P(y,z)$, we again get chiral cancellations to obtain

$$Q(x,y) P(y,z) = -\sum_{s=\pm} \overline{\lambda_s} F_s \; \chi_L \,\not{g}(x,z)\, \chi_L \,\not{g}(z,y) \;\oplus\; 0 = 0 \,.$$

Similarly, the pointwise product $P(x,y)\,Q(y,z)$ also vanishes, showing that the EL equations $[P,Q]$ are indeed satisfied in the vacuum.

Before going on, we remark for clarity that in [4], the chiral regularization ansatz (2.1) was overridden on the large scale in order to arrange a suitable normalization of the chiral fermionic states (see [4, Appendix C]). More precisely, P^N_ε was constructed by projecting out half of the states of a Dirac sea of mass m. The formula (2.1) was recovered in the limit $m \searrow 0$. In this so-called *singular mass limit*, the normalization integrals did not converge, making it possible to arrange a proper normalization, although for the limit (2.1) the normalization integral would vanish due to chiral cancellations. However, in [4, §C.1] it was explained that the formalism of the continuum limit is well-behaved in the singular mass limit, thus justifying why we were allowed to describe the regularized chiral Dirac seas by (2.1).

2.2. **Instability of the Naively Regularized Neutrino Sector.** We now give an argument which shows that if the neutrino sector is regularized in the neutrino sector according to (2.1), the system (1.5) cannot be an absolute minimum of the causal action principle (1.3). Suppose conversely that a fermionic projector P^ε, which in the neutrino sector is regularized according to (2.1), is an absolute minimum of the action principle (1.3). Then any variation of the fermionic projector can only increase the action. Evaluating this condition for specific variations leads to the notion of *state stability*, which we now recall (for details see [4, §5.6] or [12]). This notion makes it necessary to assume that our regularization is *macroscopic away from the light cone*, meaning that the difference $P^\varepsilon(x,y) - P(x,y)$ should be small pointwise except if the vector $y-x$ is close to the light cone (see [4, §5.6]). This condition seems to be fulfilled for any reasonable regularization, and thus we shall always assume it from now on. Suppose that the state Ψ is occupied by a particle (i.e. that Ψ lies in the image of the operator P^ε), whereas the state Φ is not occupied. We assume that Ψ and Φ are suitably normalized and negative definite with respect to the indefinite inner product

$$<\psi|\phi> = \int_M \overline{\psi(x)} \phi(x)\, d^4x\,. \tag{2.3}$$

Then the ansatz

$$\delta P^\varepsilon(x,y) = \Psi(x) \overline{\Psi(y)} - \Phi(x) \overline{\Phi(y)} \tag{2.4}$$

describes an admissible perturbation of P^ε. Since the number of occupied states is very large, δP^ε is a very small perturbation (which even becomes infinitesimally small in the infinite volume limit). Thus we may consider δP as a first order variation and treat the constraint in (1.3) with a Lagrange multiplier. We point out that the set of possible variations δP^ε does not form a vector space, because it is restricted by additional conditions. This is seen most easily from the fact that $-\delta P^\varepsilon$ is not an admissible variation, as it does not preserve the rank of P^ε. The fact that possible variations δP^ε are restricted has the consequence that we merely get the variational *in*equality

$$\mathcal{S}_\mu[P^\varepsilon + \delta P^\varepsilon] \geq \mathcal{S}_\mu[P^\varepsilon]\,, \tag{2.5}$$

valid for all admissible variations of the form (2.4).

Next, we consider variations which are *homogeneous*, meaning that Ψ and Φ are plane waves of momenta k respectively q,

$$\Psi(x) = \hat{\Psi}\, e^{-ikx}\,, \qquad \Phi(x) = \hat{\Phi}\, e^{-iqx}\,. \tag{2.6}$$

Then both P^ε and the variation δP depend only on the difference vector $\xi = y - x$. Thus after carrying out one integral in (1.4), we obtain a constant, so that the second integral diverges. Thinking of the infinite volume limit of a system in finite 4-volume, we can remove this divergence simply by omitting the second integral. Then (2.5) simplifies to the *state stability condition*

$$\int_M \delta \mathcal{L}_\mu[A(\xi)]\, d^4\xi \geq 0\,. \tag{2.7}$$

In order to analyze state stability for our system (1.5), we first choose the Lagrange multiplier according to (2.2). Moreover, we assume that Ψ is a state of the charged sector, whereas Φ is in the neutrino sector,

$$\hat{\Psi} = 0 \oplus \hat{\Psi}^C\,, \qquad \hat{\Phi} = \hat{\Phi}^N \oplus 0\,. \tag{2.8}$$

Since Ψ should be an occupied state, it must clearly be a solution of one of the Dirac equations $(i\partial\!\!\!/ - m_\alpha)\Psi = 0$ with $\alpha \in \{1,2,3\}$. The state Φ, on the other hand, should be unoccupied; we assume for simplicity that its momentum q is outside the support of P^N_ε,

$$q \notin \operatorname{supp} \hat{g} \tag{2.9}$$

(where $\hat{g}$ is the Fourier transform of the vector field g in (2.1)). Thus our variation removes a state from a Dirac sea in the charged sector and occupies instead an unoccupied state in the neutrino sector with arbitrary momentum q (in particular, Φ does not need to satisfy any Dirac equation). Let us compute the corresponding variation of the Lagrangian. First, using that the spectral weight is additive on direct sums, we find that

$$\begin{aligned}
\delta \mathcal{L}_{\frac{1}{4}} &= \delta\Big(|A^2| - \frac{1}{4}|A|^2\Big) = \delta|A^2| - \frac{1}{2}\,|A|\,\delta|A| \\
&= \delta\big|(A^C)^2\big| + \delta\big|(A^N)^2\big| - \frac{1}{2}\left(|A^C| + |A^N|\right)\left(\delta|A^C| + \delta|A^N|\right)\,.
\end{aligned}$$

This formula simplifies if we use that A^N vanishes due to chiral cancellations. Moreover, the first order variation of $(A^N)^2$ vanishes because

$$\delta\left((A^N)^2\right) = (\delta A^N)A^N + A^N(\delta A^N) = 0\,.$$

Finally, $\delta|A^N| = |(A^N + \delta A^N)| - |A^N| = |\delta A^N|$. This gives

$$\delta \mathcal{L}_{\frac{1}{4}} = \delta\Big(\big|(A^C)^2\big| - \frac{1}{4}\,|A^C|^2\Big) - \frac{1}{2}\,|A^C|\,|\delta A^N|\,. \tag{2.10}$$

Note that Ψ only affects the first term, whereas Φ influences only the second term. In the first term the neutrino sector does not appear, and thus the state stability analysis for one sector as carried out in [4, §5.6] and [12] applies. From this analysis, we know that the charged sector should be regularized in compliance with the condition of a distributional $\mathcal{M}P$-product (see also [6]). Then the first term in (2.10) leads to a finite variation of our action. The point is that the second term in (2.10) is *negative*. In the next lemma we show that it is even unbounded below, proving that our system indeed violates the state stability condition (2.7).

Lemma 2.1. *Suppose that P^ε is a regularization of the distribution* (1.5) *which is macroscopic away from the light cone and which in the neutrino sector is of the form* (2.1). *Then for any constant $C > 0$ there is a properly normalized, negative*

definite wave function Φ *satisfying* (2.6), (2.8) *and* (2.9) *such that the corresponding variation of the fermionic projector*

$$\delta P^\varepsilon(x,y) = -\Phi(x)\overline{\Phi(y)} \tag{2.11}$$

satisfies the inequality

$$\int_M |A^C|\,|\delta A^N|\,d^4\xi > C\,.$$

Proof. For convenience, we occupy two fermionic states of the same momentum q such that

$$\delta P^N_\varepsilon(x,y) = (\not{p}+m)\,e^{-iq(y-x)}\,, \tag{2.12}$$

where p is a vector on the lower hyperboloid $\mathcal{H}_m := \{p \,|\, p^2 = m^2 \text{ and } p^0 < 0\}$, and m is a positive parameter which involves the normalization constant. For this simple ansatz one easily verifies that the image of δP^N is indeed two-dimensional and negative definite. By occupying the two states in two separate steps, one can decompose (2.12) into two variations of the required form (2.11). Therefore, it suffices to prove the lemma for the variation (2.12).

Using (2.1) and (2.12), the variation of A^N is computed to be

$$\delta A^N = \chi_L \not{g}(x,y)(\not{p}+m)\,e^{iq\xi} + \chi_R(\not{p}+m)\,\not{g}(y,x)\,e^{-iq\xi}\,.$$

To simplify the notation, we omit the arguments x and y and write $g(\xi) = g(x,y)$. Then g is a complex vector field with $\overline{g(\xi)} = g(y,x)$. Using that our regularization is macroscopic away from the light cone, there clearly is a set $\Omega \subset M$ of positive Lebesgue measure such that both the vector field g and the function $|A^C|$ are non-zero for all $\xi \in \Omega$. Then we can choose a past directed null vector $\mathfrak{n}$ such that $\langle \mathfrak{n}, g\rangle$ is non-zero on a set $\Omega' \subset \Omega$ again of positive measure. We now consider a sequence of vectors $p_l \in \mathcal{H}_m$ which converge to the ray $\mathbb{R}^+\mathfrak{n}$ in the sense that there are coefficients c_l with

$$p_l - c_l\,\mathfrak{n} \to 0 \quad \text{and} \quad c_l \to \infty\,.$$

Then on Ω', the inner product $\langle p_l, g\rangle$ diverges as $l \to \infty$. A short computation shows that in this limit, the eigenvalues of the matrix δA^N_l also diverge. Computing these eigenvalues asymptotically, one finds that

$$|\delta A^N_l| \geq 4\,|\langle p_l, g\rangle| + \mathcal{O}(l^0)\,.$$

Hence for large l,

$$\int_M |A^C|\,|\delta A^N_l| \geq \int_{\Omega'} |A^C|\,|\langle p_l, g\rangle| \;\xrightarrow{l\to\infty}\; \infty\,,$$

completing the proof. ∎

It is remarkable that the above argument applies independent of any regularization details. We learn that regularizing the neutrino sector by a left-handed function (2.1) necessarily leads to an instability of the vacuum. The only way to avoid this instability is to consider more general regularizations where P^N_ε also involves a right-handed component.

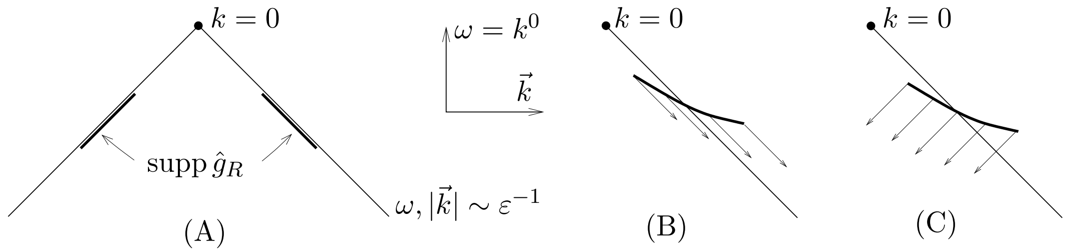

FIGURE 1. Plots of $\hat{g}_R$ exemplifying different regularization mechanisms in the neutrino sector.

2.3. **Regularizing the Vacuum Neutrino Sector – Introductory Discussion.** We begin by explaining our regularization method for one massless left-handed Dirac sea,

$$P(x,y) = \chi_L \, P_0(x,y)$$

(several seas and massive neutrinos will be considered later in this section). Working with a left-handed Dirac sea is motivated by the fact that right-handed neutrinos have never been observed in nature. To be precise, this physical observation only tells us that there should be no right-handed neutrinos in the low-energy regime. However, on the regularization scale ε^{-1}, which is at least as large as the Planck energy E_P and therefore clearly inaccessible to experiments, there might well be right-handed neutrinos. Thus it seems physically admissible to regularize P by

$$P^\varepsilon(x,y) = \chi_L \, \slashed{g}_L(x,y) + \chi_R \, \slashed{g}_R(x,y) \,, \tag{2.13}$$

provided that the Fourier transform $\hat{g}_R(k)$ vanishes if $|k^0| + |\vec{k}| \ll \varepsilon^{-1}$.

In order to explain the effect of such a *right-handed high-energy component*, we begin with the simplest example where $\hat{g}_R$ is supported on the lower mass cone,

$$\hat{\slashed{g}}_R(k) = 8\pi^2 \, \slashed{k} \, \hat{h}(\omega) \, \delta(k^2) \,, \tag{2.14}$$

where $\omega \equiv k^0$, and the non-negative function $\hat{h}$ is supported in the high-energy region $\omega \sim \varepsilon^{-1}$ (see Figure 1 (A)). We compute the Fourier integrals by

$$\begin{aligned}
\slashed{g}_R(\xi) &= 8\pi^2 \int \frac{d^4k}{(2\pi)^4} \, \slashed{k} \, \hat{h}(\omega) \, \delta(k^2) \, e^{ik\xi} = -8i\pi^2 \, \slashed{\partial}_\xi \int \frac{d^4k}{(2\pi)^4} \, \hat{h}(\omega) \, \delta(k^2) \, e^{ik\xi} \\
&= -2i \, \slashed{\partial}_\xi \int_{-\infty}^0 \frac{d\omega}{2\pi} \, \hat{h}(\omega) \, e^{i\omega t} \int_0^\infty p^2 \, dp \, \delta(\omega^2 - p^2) \int_{-1}^1 d\cos\vartheta \; e^{-ipr\cos\vartheta} \\
&= 2 \, \slashed{\partial}_\xi \left[\frac{1}{r} \int_{-\infty}^0 \frac{d\omega}{2\pi} \, \hat{h}(\omega) \, e^{i\omega t} \int_0^\infty p \, dp \, \delta(\omega^2 - p^2) \big(e^{-ipr} - e^{ipr} \big) \right] \\
&= -\slashed{\partial}_\xi \left[\frac{1}{r} \int_{-\infty}^0 \frac{d\omega}{2\pi} \, \hat{h}(\omega) \, e^{i\omega t} \left(e^{-i\omega r} - e^{i\omega r} \right) \right],
\end{aligned}$$

where we set $t = \xi^0$, $r = |\vec{\xi}|$ and chose polar coordinates $(p = |\vec{k}|, \vartheta, \varphi)$. This gives the simple formula

$$\slashed{g}_R(\xi) = -\slashed{\partial}_\xi \, \frac{h(t-r) - h(t+r)}{r} \,,$$

where h is the one-dimensional Fourier transform of $\hat{h}$. Under the natural assumption that the derivatives of $\hat{h}$ scale in powers of ε, the function h decays rapidly on the regularization scale. Then $\slashed{g}_R$ vanishes except if ξ is close to the light cone, so that

the regularization is *again macroscopic away from the light cone*. But the contribution (2.14) does affect the singularities on the light cone, and it is thus of importance in the continuum limit. More specifically, on the upper light cone away from the origin $t \approx r \gg \varepsilon$, we obtain the contribution

$$\begin{aligned} \not{g}_R(\xi) = &-\not{\partial}_\xi \,\frac{h(t-r)}{r} = -(\gamma^0 - \gamma^r)\, \frac{h'(t-r)}{r} + \gamma^r\, \frac{h(t-r)}{r^2} \\ &+ (\text{rapid decay in } r)\,, \end{aligned} \tag{2.15}$$

where we set $\gamma^r = (\vec{\xi}\vec{\gamma})/r$. This contribution is compatible with the formalism of the continuum limit, because it has a similar structure and the same scaling as corresponding contributions by a regularized Dirac sea (see [6], where the same notation and sign conventions are used).

Regularizing the neutrino sector of our fermionic projector (1.5) using a right-handed high-energy component has the consequence that *no chiral cancellations* occur. Hence the EL equations become

$$\sum_i \Big(|\lambda_i| - \mu \sum_l |\lambda_l| \Big) \frac{\overline{\lambda_i}}{|\lambda_i|}\, F_i\, P(x,y) = 0\,, \tag{2.16}$$

where i labels the eigenvalues of A_{xy}. For these equations to be satisfied, we must choose

$$\mu = \frac{1}{8}\,, \tag{2.17}$$

and furthermore we must impose that the eigenvalues of A_{xy} all have the same absolute values in the sense that

$$\big(|\lambda_i| - |\lambda_j| \big) \frac{\overline{\lambda_i}}{|\lambda_i|}\, F_i\, P(x,y) = 0 \qquad \text{for all } i,j.$$

In simple terms, the matrix A^N must have the *same spectral properties* as A^C.

This consideration points to a shortcoming of the regularization (2.14). Namely, the expression (2.15) does not involve a mass parameter, and thus the corresponding contribution to the closed chain A^N cannot have the same spectral properties as A^C, which has a non-trivial mass expansion. A possible solution to this problem is to consider states on a more *general hypersurface*, as we now explain again in the example of a spherically symmetric regularization. We choose

$$\hat{\not{g}}_R(k) = -4\pi^2\, (\gamma^0 + \gamma^k)\, \hat{h}(\omega)\, \delta\big(|\vec{k}| - K(\omega)\big)\,, \tag{2.18}$$

where $\gamma^k = \vec{k}\vec{\gamma}/k$, and h is chosen as in (2.14). We again assume that $\hat{\not{g}}$ is supported in the high-energy region, meaning that

$$\hat{h}(\omega) = 0 \qquad \text{if } |\omega| \ll \varepsilon^{-1}\,. \tag{2.19}$$

Setting $K = -\omega$, we get back to (2.14); but now the function K gives a more general dispersion relation (see Figure 1 (B)). Carrying out the Fourier integrals, we obtain

$$\begin{aligned}
g^0_R(\xi) &= -4\pi^2 \int \frac{d^4k}{(2\pi)^4}\, \hat{h}(\omega)\, \delta\big(|\vec{k}| - K(\omega)\big)\, e^{ik\xi} \\
&= -\int_{-\infty}^0 \frac{d\omega}{2\pi}\, \hat{h}(\omega)\, e^{i\omega t} \int_0^\infty p^2 dp\, \delta(p - K(\omega)) \int_{-1}^1 d\cos\vartheta\; e^{-ipr\cos\vartheta} \\
&= -\frac{i}{r} \int_{-\infty}^0 \frac{d\omega}{2\pi}\, \hat{h}(\omega)\, e^{i\omega t} \int_0^\infty p\, dp\, \delta(p - K(\omega))\, \big(e^{-ipr} - e^{ipr}\big) \\
&= \frac{i}{r} \int_{-\infty}^0 \frac{d\omega}{2\pi}\, \hat{h}(\omega)\, K(\omega)\, e^{i\omega t} \left(e^{iKr} - e^{-iKr}\right) \\
(\vec{\gamma}\vec{g}_R)(\xi) &= -4\pi^2\, (i\vec{\gamma}\vec{\nabla}) \int \frac{d^4k}{(2\pi)^4}\, \hat{h}(\omega)\, \delta(k - K(\omega))\, \frac{1}{|\vec{k}|}\, e^{ik\xi} \\
&= -\vec{\gamma}\vec{\nabla} \left[\frac{1}{r} \int_{-\infty}^0 \frac{d\omega}{2\pi}\, \hat{h}(\omega)\, e^{i\omega t} \left(e^{iKr} - e^{-iKr}\right) \right] \\
&= -\frac{i\gamma^r}{r} \int_{-\infty}^0 \frac{d\omega}{2\pi}\, \hat{h}(\omega)\, K(\omega)\, e^{i\omega t} \left(e^{iKr} + e^{-iKr}\right) \\
&\quad + \frac{\gamma^r}{r^2} \int_{-\infty}^0 \frac{d\omega}{2\pi}\, \hat{h}(\omega)\, e^{i\omega t} \left(e^{iKr} - e^{-iKr}\right).
\end{aligned}$$

Evaluating as in (2.15) on the upper light cone away from the origin, we conclude that

$$\begin{aligned}
\not{g}_R(\xi) = &\int_{-\infty}^0 \frac{d\omega}{2\pi}\, \hat{h}(\omega) \left(i\, \frac{\gamma^0 - \gamma^r}{r}\, K(\omega) + \frac{\gamma^r}{r^2} \right) e^{i(\omega t + Kr)} \\
&+ (\text{rapid decay in } r)\,.
\end{aligned} \tag{2.20}$$

For ease in notation, from now on we will omit the rapidly decaying error term. Rearranging the exponentials, we obtain

$$\not{g}_R(\xi) = \int_{-\infty}^0 \frac{d\omega}{2\pi}\, e^{i(\omega + K)r}\, \hat{h}(\omega) \left(i\, \frac{\gamma^0 - \gamma^r}{r}\, K(\omega) + \frac{\gamma^r}{r^2} \right) e^{i\omega(t-r)}\,.$$

Now the *mass expansion* can be performed by expanding the factor $\exp(i(\omega + K)r)$,

$$\begin{aligned}
\not{g}_R(\xi) &= \sum_{n=0}^\infty \frac{(ir)^n}{n!} \int_{-\infty}^0 \frac{d\omega}{2\pi}\, \hat{h}(\omega)\, (\omega + K)^n \left(i\, \frac{\gamma^0 - \gamma^r}{r}\, K(\omega) + \frac{\gamma^r}{r^2} \right) e^{i\omega(t-r)} \qquad (2.21) \\
&= \int_{-\infty}^0 \frac{d\omega}{2\pi}\, \hat{h}(\omega) \left(\frac{\gamma^r}{r^2} + i\, \frac{K\gamma^0 + \omega\gamma^r}{r} + \cdots \right) e^{i\omega(t-r)}\,.
\end{aligned}$$

We conclude that the general ansatz (2.18) gives rise to a mass expansion which is similar to that for a massive Dirac sea (see [4, Chaper 4]). By modifying the geometry of the hypersurface $\{|\vec{k}| = K(\omega)\}$, we have a lot of freedom to modify the contributions to the mass expansion. We point out that, in contrast to the mass expansion for a massive Dirac sea, the mass expansion in (2.21) involves *no logarithmic poles*. This is because here we only consider high-energy states (2.19), whereas the logarithmic poles are a consequence of the low-frequency behavior of the massive Dirac seas (for details see the discussion of the logarithmic mass problem in [4, §2.5 and §4.3]).

We now come to another regularization effect. The regularizations (2.14) and (2.18) considered so far have the property that $\not{g}_R$ is a multiple of the matrix $\chi_L(\gamma^0+\gamma^k)$, as is indicated in Figure (1) (B) by the arrows (to avoid confusion with the signs, we note that on the lower mass shell, $\not{k} = \omega\gamma^0 - \vec{k}\vec{\gamma} = \omega\,(\gamma^0+\gamma^k))$. Clearly, we could also have flipped the sign of γ^k, i.e. instead of (2.18),

$$\hat{\not{g}}_R(k) = -4\pi^2\,(\gamma^0-\gamma^k)\,\hat{h}(\omega)\,\delta\big(|\vec{k}|-K(\omega)\big) \tag{2.22}$$

(see Figure 1 (C)). In order to explain the consequence of this sign change in the simplest possible case, we consider the two functions

$$\hat{\not{g}}_\pm(k) = 8\pi^2\,\omega\,(\gamma^0\pm\gamma^k)\,\hat{h}(\omega)\,\delta(k^2)\;,$$

whose Fourier transforms are given in analogy to (2.15) on the upper light cone by

$$\not{g}_\pm(\xi) = -(\gamma^0\mp\gamma^r)\,\frac{h'(t-r)}{r} \mp \gamma^r\,\frac{h(t-r)}{r^2}\;. \tag{2.23}$$

When multiplying $\not{g}_+$ by itself, the identity $(\gamma^0+\gamma^r)^2=0$ gives rise to a cancellation. For example, in the expression

$$\frac{1}{4}\,\mathrm{Tr}\left(\not{g}_+(\xi)\,\not{g}_+(\xi)^*\right) = \frac{2\mathrm{Re}(h'(t-r)\,\overline{h(t-r)})}{r^3} - \frac{|h(t-r)|^2}{r^4} \tag{2.24}$$

the term $\sim r^{-2}$ has dropped out. The situation is different if we multiply $\not{g}_+$ by $\not{g}_-$. For example, in

$$\frac{1}{4}\,\mathrm{Tr}\left(\not{g}_+(\xi)\,\not{g}_-(\xi)^*\right) = \frac{2|h'(t-r)|^2}{r^2} - \frac{2i\,\mathrm{Im}(h'(t-r)\,\overline{h(t-r)})}{r^3} + \frac{|h(t-r)|^2}{r^4} \tag{2.25}$$

no cancellation occurs, so that the term $\sim r^{-2}$ is present. From this consideration we learn that by flipping the sign of γ^r as in (2.22), we can generate terms in the closed chain which have a different scaling behavior in the radius.

In order to clarify the last construction, it is helpful to describe the situation in terms of the general notions introduced in [4, §4.4]. The fact that the leading term in (2.15) is proportional to $(\gamma^0-\gamma^r)$ can be expressed by saying that the *vector component is null on the light cone*. When forming the closed chain, the term quadratic in the leading terms drops out, implying that $A_{xy}\sim r^{-3}$. In momentum space, this situation corresponds to the fact that the vector $\hat{g}(k)$ points almost in the same direction as k. In other words, the *shear of the surface states* is small. Thus in (2.14) and (2.18) as well as in g_+, the shear is small, implying that the vector component is null on the light cone, explaining the cancellation of the term $\sim r^{-2}$ in (2.24). The states in (2.22) and g_-, however, have a large shear. Thus the corresponding vector component is not null on the light cone, explaining the term $\sim r^{-2}$ in (2.25). We point out that states of large shear have never been considered before, as in [4] we always assumed the shear to be small. For simplicity, we refer to the states in (2.22) and g_- as *shear states*.

We next outline how the above considerations can be adapted to the general ansätze (1.8) and (1.9). In order to describe *several chiral Dirac seas*, one simply adds regularized Dirac seas, each of which might involve a right-handed high-energy component and/or shear states. In other words, in the chiral ansatz (1.8) one replaces each summand by a Dirac sea regularized as described above. In the massive ansatz (1.9), we regularize every massive Dirac sea exactly as described in [4, Chapter 4]. Moreover, in order to distinguish the neutrino sector from a massive sector, we add one or

several right-handed high-energy contributions. In this way, the regularization breaks the chiral symmetry.

We finally make a few remarks which clarify our considerations and bring them into the context of previous work.

Remark 2.2. (1) We point out that the above assumption of spherical symmetry was merely a technical simplification. But this assumption is not crucial for the arguments, and indeed it will be relaxed in §2.5. We also point out that in all previous regularizations, the occupied states formed a hypersurface in momentum space. In this paper, we will always restrict attention to such *surface states* (see [4, §4.3]). The underlying guiding principle is that one should try to build up the regularized fermionic projector with as few occupied states as possible. This can be understood from the general framework of causal variational principles as introduced in [5, 8]. Namely, in this framework the minimum of the action decreases if the number of particles gets larger[1]. Thus to construct minimizers, one should always keep the number of particles fixed. Conversely, one could also construct minimizers by keeping the action fixed and decreasing the number of particles. With this in mind, a regularization involving fewer particles corresponds to a smaller action and is thus preferable.

(2) It is worth mentioning that in all the above regularizations we worked with *null states*, meaning that for every k, the image of the operator $\hat{P}(k)$ is null with respect to the spin scalar product. Such null states can be obtained from properly normalized negative definite states by taking a singular mass limit, similar as worked out in [4, Appendix C].

(3) At first sight, our procedure for regularizing might seem very special and ad-hoc. However, it catches all essential effects of more general regularizations, as we now outline. First, states of large shear could be used just as well for the regularization of massive Dirac seas, also in the charged sector. However, our analysis in Section 6 will reveal that the EL equations will only involve the difference in the regularization used in the charged sector compared to that in the neutrino sector. Thus it is no loss in generality to regularize the charged sector simply according to [4, Chapter 4], and to account for shear states only in the neutrino sector. Next, in the high-energy region one could also work with *massive states*. In order to break the chiral symmetry, one could project out one spin state with the ansatz

$$\not{g}(p) = \frac{1}{2}\,(\mathbb{1} - \rho\not{q})\,(\not{k} + m)\,\hat{h}(k) \tag{2.26}$$

with $p^2 = m^2, q^2 = -1$ and $\langle q, k\rangle = 0$ (see [4, eq. (C.1.5)], where a corresponding Dirac sea is considered before taking the singular mass limit). However, this procedure would have two disadvantages. First, massive states would yield additional contributions to the fermionic projector, whereas (2.26) even gives rise to bilinear and pseudoscalar contributions, which would all cause technical

[1] To be precise, this results holds for operators in the class $\mathcal{P}^f$ (see [5, Def. 2.7]) if the fermionic operator is rescaled such that its trace is independent of f. In the formulation with local correlation matrices (see [8, Section 3.2]) and under the trace constraint, the canonical embedding $\mathbb{C}^f \hookrightarrow \mathbb{C}^{f+1}$ allows one to regard a system of f particles as a special system of $f+1$ particles. Since varying within the set of $f+1$-particle systems gives more freedom, it is obvious that the action decreases if f gets larger.

complications. Secondly, massive states involve both left- and right-handed components, which are coupled together in such a way that it would be more difficult to introduce a general interaction. Apart from these disadvantages, working with massive states does not seem to lead to any interesting effects. This is why we decided not to consider them in this paper.

(4) We mention that for a fully convincing justification of the vacuum fermionic projector (1.5) and of our regularization method, one should extend the *state stability analysis* from [12] to a system of a charged sector and a neutrino sector. Since this analysis only takes into account the behavior of the fermionic projector away from the light cone, the high-energy behavior of P^ε plays no role, so that one could simply work with the explicit formula for the unregularized fermionic projector (1.5). Then the methods of [12] apply to each of the sectors. However, the two sectors are coupled by the term $|A|^2$ in the Lagrangian. The results of this analysis will depend on the value of the Lagrange multiplier (2.17) as well as on the choice of all lepton masses (including the neutrino masses). Clearly, the details of this analysis are too involved for predicting results. For the moment, all one can say is that there is no general counter argument (in the spirit of §2.2) which might prevent state stability.

2.4. **Ruling out the Chiral Neutrino Ansatz.** In this section, we give an argument which shows that for chiral neutrinos there is no regularization which gives rise to a stable minimum of the causal action principle. More precisely, we will show that even taking into account the regularization effects discussed in the previous section, it is impossible to arrange that the vacuum satisfies the EL equations in the continuum limit (2.16) and (2.17). Our argument applies in such generality (i.e. without any specific assumptions on the regularization) that it will lead us to drop the ansatz of chiral neutrinos (1.8), leaving us with the ansatz of massive neutrinos (1.9).

Considering massive neutrinos is clearly consistent with the experimental observation of neutrino oscillations. Based on these experimental findings, we could also have restricted attention to the ansatz (1.9) right away. On the other hand, considering also chiral neutrinos (1.8) has the advantage that we can conclude that massive neutrinos are needed even for mathematical consistency. This conclusion is of particular interest because in the neutrino experiments, the mass of the neutrinos is observed indirectly from the fact that different generations of neutrinos are converted into each other. This leaves the possibility that neutrinos might be massless, and that the neutrino oscillations can be explained instead by modifying the weak interaction. The following argument rules out this possibility by giving an independent reason why there must be massive neutrinos.

Recall that the Dirac seas in the charged sector P^C, (1.6), can be written as

$$P_m(x,y) \;=\; (i\partial\!\!\!/_x + m)\, T_{m^2}(x,y)\;, \tag{2.27}$$

where T_{m^2} is the Fourier transform of the lower mass shell,

$$T_{m^2}(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2-m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\;.$$

Computing this Fourier integral and expanding the resulting Bessel functions gives the expansion in position space

$$\begin{aligned} T_{m^2}(x,y) = &-\frac{1}{8\pi^3} \left(\frac{\mathrm{PP}}{\xi^2} \;+\; i\pi\delta(\xi^2)\,\varepsilon(\xi^0) \right) \\ &+\; \frac{m^2}{32\pi^3} \left(\log|m^2\xi^2| + c + i\pi\,\Theta(\xi^2)\,\epsilon(\xi^0) \right) + \mathcal{O}(\xi^2\,\log(\xi^2))\,. \end{aligned} \tag{2.28}$$

(see [4, §2.5] or [7, §4.4]). The point for what follows is that the light-cone expansion of $P_m(x,y)$ involves a logarithmic pole $\sim \log(\xi^2)$. As a consequence, in the EL equations (2.16) we get contributions to (2.16) which involve the logarithm of the radius $|\vec{x}-\vec{y}|$ (for details see [7, § 5.1] or the weak evaluation formula (2.32) below). In order to satisfy the EL equations, these logarithmic contributions in the charged sector must be compensated by corresponding logarithmic contributions in the neutrino sector.

Now assume that we consider the chiral neutrino ansatz (1.8). Then the light-cone expansion of T_N does not involve logarithmic poles (indeed, the distribution P_0 can be given explicitly in position space by taking the limit $m \searrow 0$ in (2.27) and (2.28)). Thus the logarithmic contributions in the radius must come from the high-energy component to the fermionic projector. However, as one sees explicitly from the formulas (2.21) and (2.23), the high-energy component is a Laurent series in the radius and does not involve any logarithms. This explains why with chiral neutrinos alone it is impossible to satisfy the EL equations.

This problem can also be understood in more general terms as follows. The logarithmic poles of $P_m(x,y)$ are an infrared effect related to the fact that the square root is not an analytic function (see the discussion of the so-called logarithmic mass problem in [4, §2.5 and §4.5]). Thus in order to arrange logarithmic contributions in the high-energy region, one would have to work with states on a surface with a singularity. Then the logarithm in the radius would show up in the next-to leading order on the light cone. Thus in order to compensate the logarithms in (2.28), the contribution by the high-energy states would be just as singular on the light cone as the contribution by the highest pole in (2.28). Apart from the fact that it seems difficult to construct such high-energy contributions, such constructions could no longer be regarded as regularizations of Dirac sea structures. Instead, one would have to put in specific additional structures ad hoc, in contrast to the concept behind the method of variable regularization (see [4, §4.1]).

The above arguments show that at least one generation of neutrinos must be massive. In particular, we must give up the ansatz (1.8) of chiral neutrinos. Instead, we shall always work with massive neutrinos (1.9), and we need to assume that at least one of the masses $\tilde{m}_\beta$ is non-zero.

For clarity, we finally remark that our arguments also leave the possibility to choose another ansatz which involves a combination of both chiral and massive neutrinos, i.e.

$$P^N(x,y) = \sum_{\beta=1}^{\beta_0} \chi_L P_0(x,y) + \sum_{\beta=\beta_0+1}^{3} P_{m_\beta}(x,y) \qquad \text{with} \qquad \beta_0 \in \{1,2\}\,. \tag{2.29}$$

The only reason why we do not consider this ansatz here is that it seems more natural to describe all neutrino generations in the same way. All our methods could be extended in a straightforward way to the ansatz (2.29).

2.5. **A Formalism for the Regularized Vacuum Fermionic Projector.** In the following sections §2.5 and §2.6, we incorporate the regularization effects discussed in §2.3 to the formalism of the continuum limit. Beginning with the vacuum, we recall that in [4, §4.5] we described the regularization by complex factors $T^{(n)}_{[p]}$ and $T^{(n)}_{\{p\}}$ (see also [7, §5.1]). The upper index n tells about the order of the singularity on the light cone, whereas the lower index keeps track of the orders in a mass expansion. In §2.3, we considered a chiral decomposition (2.13) and chose the left- and right-handed components independently. This can be indicated in our formalism by a chiral index $c \in \{L, R\}$, which we insert into the subscript. Thus we write the regularization (2.13) and (2.14) symbolically as

$$P^\varepsilon(x,y) = \frac{i}{2} \left(\chi_L \, \not{\xi} T^{(-1)}_{[L,0]} + \chi_R \, \not{\xi} T^{(-1)}_{[R,0]} \right).$$

If the regularization effects of the previous section are *not* used in the left- or right-handed component, we simply omit the chiral index. Thus if we work with general surface states or shear states only in the right-handed component, we leave out the left-handed chiral index,

$$P^\varepsilon(x,y) = \frac{i}{2} \left(\chi_L \, \not{\xi} T^{(-1)}_{[0]} + \chi_R \, \not{\xi} T^{(-1)}_{[R,0]} \right).$$

When using the same notation as in the charged sector, we always indicate that we assume the corresponding regularizations to be compatible. Thus for factors $T^{(n)}_\circ$ without a chiral index, we shall use the same calculation rules in the neutrino and in the charged sector. This will also make it possible to introduce an interaction between these sectors (for details see §2.6 and Appendix A). If we consider a sector of massive neutrinos (1.9), we first perform the mass expansion of every Dirac sea

$$P^\varepsilon_m = \frac{i\not{\xi}}{2} \sum_{n=0}^\infty \frac{m^{2n}}{n!} \, T^{(-1+n)}_{[2n]} + \sum_{n=0}^\infty \frac{m^{2n+1}}{n!} \, T^{(n)}_{[2n+1]} \tag{2.30}$$

and then add the chiral index to the massless component,

$$\begin{aligned} P^\varepsilon_m(x,y) &= \frac{i}{2} \left(\chi_L \, \not{\xi} T^{(-1)}_{[0]} + \chi_R \, \not{\xi} T^{(-1)}_{[R,0]} \right) \\ &\quad + \frac{i\not{\xi}}{2} \sum_{n=1}^\infty \frac{m^{2n}}{n!} \, T^{(-1+n)}_{[2n]} + \sum_{n=0}^\infty \frac{m^{2n+1}}{n!} \, T^{(n)}_{[2n+1]} \,. \end{aligned} \tag{2.31}$$

Now the regularization effects of the previous section can be incorporated by introducing more general factors $T^{(n)}_{[c,p]}$ and $T^{(n)}_{\{c,p\}}$ and by imposing suitable computation rules. Before beginning, we point out that the more general factors should all comply with our weak evaluation rule

$$\int_{|\vec{\xi}|-\varepsilon}^{|\vec{\xi}|+\varepsilon} dt \; \eta(t,\vec{\xi}) \, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ \, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ \, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \;=\; \eta(|\vec{\xi}|, \vec{\xi}) \, \frac{c_{\text{reg}}}{(i|\vec{\xi}|)^L} \, \frac{\log^k(\varepsilon|\vec{\xi}|)}{\varepsilon^{L-1}} \,, \tag{2.32}$$

which holds up to

$$\text{(higher orders in } \varepsilon/\ell_{\text{macro}} \text{ and } \varepsilon/|\vec{\xi}|) \,. \tag{2.33}$$

Here L is the degree defined by $\deg T^{(n)}_\circ = 1-n$, and c_{reg} is a so-called *regularization parameter* (for details see again [4, §4.5] or [7, §5.1]). The quotient of products of

factors $T^{(n)}_\circ$ and $\overline{T^{(n)}_\circ}$ in (2.32) is referred to as a *simple fraction*. In order to take into account the mass expansion (2.21), we replace every factor $T^{(-1)}_{[c,0]}$ by the formal series

$$\sum_{n=0}^\infty \frac{1}{n!}\,\frac{1}{\delta^{2n}} T^{(-1+n)}_{[c,2n]}\:. \tag{2.34}$$

This notation has the advantage that it resembles the even part of the standard mass expansion (2.30). In order to get the scaling dimensions right, we inserted a factor δ^{-2n}, where the parameter δ has the dimension of a length. The scaling of δ will be specified later (see (4.21), §4.6 and Section 8). For the moment, in order to make sense of the mass expansion, we only need to assume that the

$$\text{length scale } \delta \gg \varepsilon\:. \tag{2.35}$$

But δ could be much smaller than the Compton wave length of the fermions of the system. It could even be on the same scale as the regularization length ε. We thus replace (2.31) by

$$\begin{aligned} P^\varepsilon_m(x,y) &= \chi_L\, \frac{i\not{\xi}}{2}\, T^{(-1)}_{[0]} + \chi_R\, \frac{i\not{\xi}}{2} \sum_{n=0}^\infty \frac{1}{n!}\,\frac{1}{\delta^{2n}} T^{(-1+n)}_{[R,2n]} \\ &\quad + \frac{i\not{\xi}}{2} \sum_{n=1}^\infty \frac{m^{2n}}{n!}\, T^{(-1+n)}_{[2n]} + \sum_{n=0}^\infty \frac{m^{2n+1}}{n!}\, T^{(n)}_{[2n+1]}\:. \end{aligned} \tag{2.36}$$

The effect of large shear can be incorporated in our *contraction rules*, as we now explain. Recall that our usual contraction rules read

$$(\not{\xi}^{(n)}_{[p]})^j\, (\not{\xi}^{(n')}_{[p']})_j = \frac{1}{2}\left(z^{(n)}_{[p]} + z^{(n')}_{[p']}\right) + \big(\text{higher orders in } \varepsilon/|\vec{\xi}|\big) \tag{2.37}$$

$$z^{(n)}_{[p]}\, T^{(n)}_{[p]} = -4\left(n\, T^{(n+1)}_{[p]} + T^{(n+2)}_{\{p\}}\right) \tag{2.38}$$

(and similarly for the complex conjugates, cf. [4, §4.5] or [7, §5.1]). We extend the first rule in the obvious way by inserting lower chiral indices. In the second rule we insert a factor δ^{-2},

$$z^{(n)}_{[c,p]}\, T^{(n)}_{[c,p]} = -4\left(n\, T^{(n+1)}_{[c,p]} + \frac{1}{\delta^2}\, T^{(n+2)}_{\{c,p\}}\right). \tag{2.39}$$

The factor δ^{-2} has the advantage that it ensures that the factors with square and curly brackets have the same scaling dimension (as one sees by comparing (2.39) with (2.34) or (2.30); we remark that this point was not taken care of in [4] and [7], simply because the factors with curly brackets played no role). The term $\delta^{-2} T^{(n+2)}_{\{c,p\}}$ can be associated precisely to the shear states. For example, in the expression

$$\frac{1}{8}\,\mathrm{Tr}\left((\not{\xi} T^{(-1)}_{[0]})\,(\not{\xi} T^{(-1)}_{[R,0]})\right) = T^{(0)}_{[0]} T^{(-1)}_{[R,0]} + T^{(-1)}_{[0]} T^{(0)}_{[R,0]} - T^{(1)}_{\{0\}} T^{(-1)}_{[R,0]} - \frac{1}{\delta^2}\, T^{(-1)}_{[0]} T^{(1)}_{\{R,0\}}\:,$$

the last summand involves an additional scaling factor of r and can thus be used to describe the effect observed in (2.25). Using again (2.35), we can reproduce the scaling of the first summand in (2.25).

In the weak evaluation formula (2.32), one can integrate by parts. This gives rise to the following *integration-by-parts rules*. On the factors $T^{(n)}_\circ$ we introduce a derivation ∇ by

$$\nabla T^{(n)}_\circ = T^{(n-1)}_\circ\:.$$

Extending this derivation with the Leibniz and quotient rules, the integration-by-parts rules states that

$$\nabla \left(\frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ \, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ \, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \right) = 0 \,. \tag{2.40}$$

As shown in [4, Appendix E], there are no further relations between the factors $T^{(a)}_\circ$.

We finally point out that the chiral factors $T^{(n)}_{[c,p]}$ and $T^{(n)}_{\{c,p\}}$ were introduced in such a way that the weak evaluation formula (2.32) remains valid. However, one should keep in mind that these chiral factors do not have logarithmic singularities on the light cone, which implies that they have no influence on the power k in (2.32). This follows from the fact that the chiral factors only describe high-energy effects, whereas the logarithmic poles are a consequence of the low-frequency behavior of the massive Dirac seas (see also the explicit example (2.21) and the explanation thereafter).

2.6. **Interacting Systems, Regularization of the Light-Cone Expansion.** We now extend the previous formalism such as to include a general interaction; for the derivation see Appendix A. For simplicity, we restrict attention to the system (1.5) with massive neutrinos (1.9) and a non-trivial regularization of the neutrino sector by right-handed high-energy states. But our methods apply to more general systems as well (see Remark 2.3 below). In preparation, as in [4, §2.3] and [7, §4.1] it is helpful to introduce the *auxiliary fermionic projector* as the direct sum of all Dirac seas. In order to allow the interaction to be as general as possible, it is preferable to describe the right-handed high-energy states by a separate component of the auxiliary fermionic projector. Thus we set

$$P^{\text{aux}} = P^N_{\text{aux}} \oplus P^C_{\text{aux}} \,, \tag{2.41}$$

where

$$P^N_{\text{aux}} = \Big(\bigoplus_{\beta=1}^3 P_{\tilde{m}_\beta} \Big) \oplus 0 \qquad \text{and} \qquad P^C_{\text{aux}} = \bigoplus_{\beta=1}^3 P_{m_\beta} \,. \tag{2.42}$$

Note that P^{aux} is composed of seven direct summands, four in the neutrino and three in the charged sector. As the fourth component of the neutrino sector is reserved for right-handed high-energy neutrinos (possibly occupying shear or general surface states), the corresponding component vanishes without regularization (2.42).

In order to recover P^{aux} from a solution of the Dirac equation, we introduce the *chiral asymmetry matrix* X by

$$X = (\mathbb{1}_{\mathbb{C}^3} \oplus \tau_{\text{reg}}\, \chi_R) \oplus \mathbb{1}_{\mathbb{C}^3} \,. \tag{2.43}$$

Here τ_{reg} is a dimensionless parameter, which we always assume to take values in the range

$$0 < \tau_{\text{reg}} \leq 1 \,.$$

It has two purposes. First, it indicates that the corresponding direct summand involves a non-trivial regularization. This will be useful below when we derive constraints for the interaction. Second, it can be used to modify the amplitude of the regularization effects. In the limit $\tau_{\text{reg}} \searrow 0$, the general surface states and shear states are absent, whereas in the case $\tau_{\text{reg}} = 1$, they have the same order of magnitude as the regular states.

Next, we introduce the *mass matrix* Y by

$$Y = \frac{1}{m} \left(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, 0, m_1, m_2, m_3 \right) \tag{2.44}$$

(here m is an arbitrary mass parameter which makes Y dimensionless and is useful for the mass expansion; see also [4, §2.3] or [7, §4.1]). In the limiting case $\tau_{\text{reg}} \searrow 0$, we can then write P^{aux} as

$$P^{\text{aux}} = X t = t X^* \qquad \text{with} \qquad t := \bigoplus_{\beta=1}^{7} P_{m Y_\beta} \,. \tag{2.45}$$

In the case $\tau_{\text{reg}} > 0$, the fourth direct summand will contain additional states. We here model these states by a massless Dirac sea (the shear, and general surface states will be obtained later from these massless Dirac states by building in a non-trivial regularization). Thus we also use the ansatz (2.45) in the case $\tau_{\text{reg}} > 0$. Since t is composed of Dirac seas, it is a solution of the Dirac equation

$$(i\partial\!\!\!/ - mY)\, t = 0 \,. \tag{2.46}$$

In order to introduce the interaction, we insert an operator $\mathcal{B}$ into the Dirac equation,

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\, \tilde{t} = 0 \,. \tag{2.47}$$

Just as explained in [4, §2.2] and [10], the *causal perturbation theory* defines $\tilde{t}$ in terms of a unique perturbation series. The *light-cone expansion* (see [4, §2.5] and the references therein) is a method for analyzing the singularities of $\tilde{t}$ near the light cone. This gives a representation of $\tilde{t}$ of the form

$$\begin{aligned} \tilde{t}(x,y) = \sum_{n=-1}^{\infty} \sum_k m^{p_k} (\text{nested bounded line integrals}) \times T^{(n)}(x,y) \\ + P^{\text{le}}(x,y) + P^{\text{he}}(x,y) \,, \end{aligned} \tag{2.48}$$

where $P^{\text{le}}(x,y)$ and $P^{\text{he}}(x,y)$ are smooth to every order in perturbation theory. The remaining problem is to insert the chiral asymmetry matrix X into the perturbation series to obtain the auxiliary fermionic projector with interaction $\tilde{P}^{\text{aux}}$. As is shown in Appendix A, the operator $\tilde{P}^{\text{aux}}$ can be uniquely defined in full generality, without any assumptions on $\mathcal{B}$. However, for the resulting light-cone expansion to involve only *bounded* line integrals, we need to assume the *causality compatibility condition*

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\, X = X^* \,(i\partial\!\!\!/ + \mathcal{B} - mY) \qquad \text{for all } \tau_{\text{reg}} \in (0,1]. \tag{2.49}$$

A similar condition is considered in [4, Def. 2.3.2]. Here the additional parameter τ_{reg} entails the further constraint that the right-handed neutrino states must not interact with the regular sea states. This constraint can be understood from the fact that gauge fields or gravitational fields should change space-time only on the macroscopic scale, but they should leave the microscopic space-time structure unchanged. This gives rise to conditions for the admissible interactions of the high-energy states. As is worked out in Appendix A, the gauge fields and the gravitational field must not lead to a "mixing" of the right-handed high-energy states with other states.

Assuming that the causality compatibility condition holds, the auxiliary fermionic projector of the sea states P^{sea} is obtained similar to (2.45) by multiplication with the chiral asymmetry matrix. Incorporating the mass expansion similar to (2.34) leads to the following formalism. We multiply the formulas of the light-cone expansion by X

from the left or by X^* from the right (which as a consequence of (2.49) gives the same result). The regularization is built in by the formal replacements

$$m^p \, T^{(n)} \to m^p \, T^{(n)}_{[p]} \,, \tag{2.50}$$

$$\tau_{\mathrm{reg}} \, T^{(n)} \to \tau_{\mathrm{reg}} \sum_{k=0}^{\infty} \frac{1}{k!} \, \frac{1}{\delta^{2k}} T^{(k+n)}_{[R,2n]} \,. \tag{2.51}$$

Next, we introduce particles and anti-particles by occupying additional states or removing states from the sea, i.e.

$$P^{\mathrm{aux}}(x,y) = P^{\mathrm{sea}}(x,y) - \frac{1}{2\pi} \sum_{k=1}^{n_p} \Psi_k(x) \overline{\Psi_k(y)} + \frac{1}{2\pi} \sum_{l=1}^{n_a} \Phi_l(x) \overline{\Phi_l(y)} \,. \tag{2.52}$$

For the normalization of the particle and anti-particle states we refer to [4, §2.8] and [7, §4.3]. Finally, we introduce the regularized fermionic projector P by taking the *partial trace* (see also [4, §2.3] or [7, eq. (4.4)]),

$$(P)^i_j = \sum_{\alpha,\beta} (\tilde{P}^{\mathrm{aux}})^{(i,\alpha)}_{(j,\beta)} \,, \tag{2.53}$$

where $i,j \in \{1,2\}$ is the sector index, whereas the indices α and β run over the corresponding generations (i.e., $\alpha \in \{1,\ldots 4\}$ if $i=1$ and $\alpha \in \{1,2,3\}$ if $i=2$). We again indicate the partial trace of the mass matrices by accents (see [4, §7.1] or [7, eq. (5.2)]),

$$\hat{Y} = \sum_\alpha Y^\alpha_\alpha \,, \qquad \acute{Y} Y \cdots \grave{Y} = \sum_{\alpha,\beta,\gamma_1,\ldots,\gamma_{p-1}} Y^\alpha_{\gamma_1} \cdots Y^{\gamma_1}_{\gamma_2} \cdots Y^{\gamma_{p-1}}_\beta . \tag{2.54}$$

Remark 2.3. *(Regularizing general systems with interaction)* We now outline how the above construction fits into a general framework for describing interacting fermion system with chiral asymmetry. Suppose we consider a system which in the vacuum is composed of a direct sum of sums of Dirac seas, some of which involve non-trivial regularizations composed of right- or left-handed high-energy shear or general surface states. Then the interaction can be introduced as follows: To obtain the auxiliary fermionic projector, we replace the sums by direct sums. For each Dirac sea which should involve a non-trivial regularization, we add a direct summand involving a left- or right-handed massless Dirac sea. After reordering the direct summands, we thus obtain

$$P^{\mathrm{aux}} = \Big(\bigoplus_{\ell=1}^{\ell_1} P_{m_j} \Big) \oplus \Big(\bigoplus_{\ell=\ell_1+1}^{\ell_2} \chi_L P_0 \Big) \oplus \Big(\bigoplus_{\ell=\ell_2+1}^{\ell_{\max}} \chi_R P_0 \Big) \tag{2.55}$$

with parameters $1 \leq \ell_1 \leq \ell_2 \leq \ell_{\max}$. In order to keep track of which direct summand belongs to which sector, we form a partition $L_1, \ldots, L_N$ of $\{1, \ldots, \ell_{\max}\}$ such that L_i contains all the seas in the i^{th} sector. Then the fermionic projector of the vacuum is obtained by taking the partial trace as follows,

$$P^i_j = \sum_{\alpha \in L_i} \sum_{\beta \in L_j} (P^{\mathrm{aux}})^\alpha_\beta \,, \qquad i,j = 1, \ldots, N \,. \tag{2.56}$$

The next step is to specify the intended form of the regularization by parameters $\tau^{\mathrm{reg}}_1, \ldots, \tau^{\mathrm{reg}}_p$ with $p \in \mathbb{N}_0$. The rule is that to every left- or right-handed massless Dirac sea which corresponds to a non-trivial regularization we associate a parameter τ^{reg}_k. Regularizations which we consider to be identical are associated the same

parameter; for different regularizations we take different parameters. Introducing the chiral asymmetry matrix X, the mass matrix Y, and the distribution t by

$$mY = (m_1, \ldots, m_{\ell_1}) \oplus (0, \ldots, 0) \oplus (0, \ldots, 0) \tag{2.57}$$

$$X = (1, \ldots, 1) \oplus \chi_L \Big(1, \ldots, 1, \tau^{\text{reg}}_{k_1}, \ldots \tau^{\text{reg}}_{k_a}\Big) \oplus \chi_R \Big(1, \ldots, 1, \tau^{\text{reg}}_{k_{a+1}}, \ldots \tau^{\text{reg}}_{k_b}\Big) \tag{2.58}$$

$$t = \Big(\bigoplus_{\ell=1}^{\ell_1} P_{m_\ell} \Big) \oplus \Big(\bigoplus_{\ell=\ell_1+1}^{\ell_2} P_0 \Big) \oplus \Big(\bigoplus_{\ell=\ell_2+1}^{\ell_{\max}} P_0 \Big), \tag{2.59}$$

the interaction can again be described by inserting an operator $\mathcal{B}$ into the Dirac equation (2.47). Now the causality compatibility condition (2.49) must hold for all values of the regularization parameters $\tau^{\text{reg}}_1, \ldots, \tau^{\text{reg}}_k$, thus allowing for an interaction only between seas with identical regularization. Using the causal perturbation expansion and the light-cone expansion, we can again represent $\tilde{t}$ in the form (2.48). The regularization is again introduced by setting $P^{\text{sea}} = tX^*$ and applying the replacement rules (2.50) as well as

$$\chi_{L/R}\, \tau^{\text{reg}}_j\, T^{(n)} \to \chi_{L/R}\, \tau^{\text{reg}}_j \sum_{k=0}^\infty \frac{1}{k!}\, \frac{1}{\delta^{2k}} T^{(k+n)}_{[R/L,2n,j]} \,,$$

where the additional index j in the subscript $[R/L, 2n, j]$ indicates that the factors $T^{(n)}_\circ$ corresponding to different parameters τ_j must be treated as different functions. This means that the basic fractions formed of these functions are all linearly independent in the sense made precise in [4, Appendix E]. Finally, we introduce particles and anti-particles again by (2.52) and obtain the fermionic projector by taking the partial trace (2.56). ◊

2.7. **The ι-Formalism.** In the formalism of the continuum limit reviewed in §2.5, the regularization is described in terms of contraction rules. While this formulation is most convenient for most computations, it has the disadvantage that the effect of the regularization on the inner factors $\xi\!\!\!/^{(n)}_\circ$ is not explicit. The ι-formalism remedies this shortcoming by providing more detailed formulas for the regularized fermionic projector in position space. The formalism will be used in §3.2, §4.6 and §5.2. It will also be important for the derivation of the Einstein equations in Section 8. Here we introduce the formalism and illustrate its usefulness in simple examples.

We begin for clarity with one Dirac sea in the charged sector. Then the mass expansion gives (cf. (2.30); see also [4, §4.5])

$$P^\varepsilon_m = \frac{i}{2} \sum_{n=0}^\infty \frac{m^{2n}}{n!}\, \xi\!\!\!/\, T^{(-1+n)}_{[2n]} + \sum_{n=0}^\infty \frac{m^{2n+1}}{n!}\, T^{(n)}_{[2n+1]} \,.$$

We choose a vector $\check{\xi}$ which is real-valued, lightlike and approximates ξ, i.e.

$$\check{\xi}^2 = 0 \,, \qquad \overline{\check{\xi}} = \check{\xi} \qquad \text{and} \qquad \check{\xi} = \xi + (\text{higher orders in } \varepsilon/|\vec{\xi}|) \,. \tag{2.60}$$

Replacing all factors ξ in P^ε_m by $\check{\xi}$, we obtain the function $\check{P}^\varepsilon_m$,

$$\check{P}^\varepsilon_m := \frac{i}{2} \sum_{n=0}^\infty \frac{m^{2n}}{n!}\, \check{\xi}\!\!\!/\, T^{(-1+n)}_{[2n]} + \sum_{n=0}^\infty \frac{m^{2n+1}}{n!}\, T^{(n)}_{[2n+1]} \,.$$

Clearly, this function differs from P^ε_m by vectorial contributions. We now want to determine these additional contributions by using that the contraction rules (2.37)

and (2.38) hold. It is most convenient to denote the involved vectors by $\iota^{(n)}_{[p]}$, which we always normalize such that

$$\langle \check{\xi}, \iota^{(n)}_\circ \rangle = 1 \,. \tag{2.61}$$

Then the contraction rules (2.37) and (2.38) are satisfied by the ansatz

$$P^\varepsilon_m = \check{P}^\varepsilon_m - i \sum_{n=0}^\infty \frac{m^{2n}}{n!} \,\not{\iota}^{(-1+n)}_{[2n]} \Big((n-1)\, T^{(n)}_{[2n]} + T^{(n+1)}_{\{2n\}} \Big) \,, \tag{2.62}$$

as is verified by a straightforward calculation. To explain the essence of this computation, let us consider only the leading contribution in the mass expansion,

$$P^\varepsilon = \frac{i}{2}\, \check{\not{\xi}} T^{(-1)}_{[0]} + i \not{\iota}^{(-1)}_{[0]}\, T^{(0)}_{[0]} + (\deg < -1) + \mathcal{O}(m) \,. \tag{2.63}$$

Taking the square, we obtain

$$\begin{aligned} (P^\varepsilon)^2 &= -\langle \check{\xi}, \iota^{(-1)}_{[0]} \rangle\, T^{(-1)}_{[0]} T^{(0)}_{[0]} - \langle \iota^{(-1)}_{[0]}, \iota^{(-1)}_{[0]} \rangle\, T^{(0)}_{[0]} T^{(0)}_{[0]} + (\deg < -2) + \mathcal{O}(m) \\ &= -T^{(-1)}_{[0]} T^{(0)}_{[0]} - \langle \iota^{(-1)}_{[0]}, \iota^{(-1)}_{[0]} \rangle\, T^{(0)}_{[0]} T^{(0)}_{[0]} + (\deg < -2) + \mathcal{O}(m) \,. \end{aligned}$$

The first summand reproduces the contraction rules (2.37) and (2.38). Compared to this first summand, the second summand is of higher order in $\varepsilon/|\vec{\xi}|$. It is thus omitted in the formalism of the continuum limit, where only the leading contribution in $\varepsilon/|\vec{\xi}|$ is taken into account (for details see [4, Chapter 4]). More generally, when forming composite expressions of (2.62) in the formalism of the continuum limit, only the mixed products $\langle \xi^{(n)}_\circ, \iota^{(n')}_\circ \rangle$ need to be taken into account, whereas the products $\langle \iota^{(n)}_\circ, \iota^{(n')}_\circ \rangle$ involving two factors $\iota^{(\cdot)}_\circ$ may be disregarded. With this in mind, one easily sees that the ansatz (2.62) indeed incorporates the contraction rules (2.37) and (2.38). Concerning the uniqueness of the representation (2.62), there is clearly the freedom to change the vectors $\iota^{(n)}_\circ$, as long as the relations (2.61) are respected. Apart from this obvious arbitrariness, the representation (2.62) is unique up to contributions of higher order in $\varepsilon/|\vec{\xi}|$, which can be neglected in a weak evaluation on the light cone.

In order to extend the above formalism to include the regularization effects in the neutrino sector, we define $\check{P}^\varepsilon_m$ by replacing all factors ξ in (2.36) by $\check{\xi}$. Writing

$$P^\varepsilon_m(x,y) = \check{P}^\varepsilon_m - i \chi_L \not{\iota}^{(-1)}_{[0]} \Big(-T^{(0)}_{[0]} + T^{(1)}_{\{0\}} \Big) \tag{2.64}$$

$$\qquad - i \chi_R \sum_{n=0}^\infty \frac{1}{n!}\, \frac{1}{\delta^{2n}}\, \not{\iota}^{(-1+n)}_{[2n]} \Big((n-1)\, T^{(n)}_{[R,2n]} + \frac{1}{\delta^2}\, T^{(n+1)}_{\{R,2n\}} \Big) \tag{2.65}$$

$$\qquad - i \sum_{n=1}^\infty \frac{m^{2n}}{n!}\, \not{\iota}^{(-1+n)}_{[2n]} \Big((n-1)\, T^{(n)}_{[2n]} + T^{(n+1)}_{\{2n\}} \Big) \,, \tag{2.66}$$

a straightforward calculation shows that the contraction rules (2.37), (2.38) and (2.39) are indeed respected.

Clearly, the ι-formalism is equivalent to the standard formalism of §2.5. However, it makes some computations more transparent, as we now explain. For simplicity, we again consider the leading order in the mass expansion (2.63) and omit all correction

terms, i.e.

$$\begin{aligned} P^\varepsilon(x,y) &= \frac{i}{2}\, \check{\slashed{\xi}} T^{(-1)}_{[0]} + i \slashed{\iota}^{(-1)}_{[0]}\, T^{(0)}_{[0]} \\ P^\varepsilon(y,x) &= P^\varepsilon(x,y)^* = -\frac{i}{2}\, \check{\slashed{\xi}} \overline{T^{(-1)}_{[0]}} - i \overline{\slashed{\iota}^{(-1)}_{[0]}\, T^{(0)}_{[0]}}\,. \end{aligned} \tag{2.67}$$

Suppose we want to compute the eigenvalues of the closed chain. As we already saw in the example (2.63), contractions between two factors $\iota^{(n)}_\circ$ are of higher order in $\varepsilon/|\vec{\xi}|$. Thus, in view of the relations (2.60), it suffices to take into account the mixed terms, i.e.

$$A_{xy} = \frac{1}{2}\, \slashed{\iota}^{(-1)}_{[0]} \check{\slashed{\xi}}\, T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} + \frac{1}{2}\, \check{\slashed{\xi}}\, \overline{\slashed{\iota}^{(-1)}_{[0]}}\; T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} + (\text{higher orders in } \varepsilon/|\vec{\xi}|)\,. \tag{2.68}$$

When taking powers of A_{xy}, any product of the first summand in (2.68) with the second summand in (2.68) vanishes, because we get two adjacent factors $\check{\slashed{\xi}}$. Similarly, we also get zero when the second summand is multiplied by the first summand, because in this case we get two adjacent factors $\slashed{\iota}$. We thus obtain

$$(A_{xy})^p = \Big(\frac{1}{2}\, \slashed{\iota}^{(-1)}_{[0]} \check{\slashed{\xi}}\, T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} \Big)^p + \Big(\frac{1}{2}\, \check{\slashed{\xi}}\, \overline{\slashed{\iota}^{(-1)}_{[0]}}\; T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} \Big)^p, \tag{2.69}$$

where we again omitted the higher orders in $\varepsilon/|\vec{\xi}|$. Moreover, powers of products of $\check{\slashed{\xi}}$ and $\slashed{\iota}$ can be simplified using the anti-commutation relations; for example,

$$\Big(\check{\slashed{\xi}}\, \overline{\slashed{\iota}^{(-1)}_{[0]}} \Big)^2 = \check{\slashed{\xi}}\, \overline{\slashed{\iota}^{(-1)}_{[0]}} \check{\slashed{\xi}} \overline{\slashed{\iota}^{(-1)}_{[0]}} = 2\, \check{\slashed{\xi}}\, \langle \overline{\iota^{(-1)}_{[0]}}, \check{\xi} \rangle\, \overline{\slashed{\iota}^{(-1)}_{[0]}}\,,$$

and applying (2.61) together with the fact that $\check{\xi}$ is real, we obtain

$$\Big(\check{\slashed{\xi}}\, \overline{\slashed{\iota}^{(-1)}_{[0]}} \Big)^2 = 2\, \check{\slashed{\xi}}\, \overline{\slashed{\iota}^{(-1)}_{[0]}}\,.$$

This shows that the Dirac matrices in (2.69) in the first and second summand in (2.69) both have the eigenvalues two and zero. From this fact we can immediately read off the eigenvalues of (2.68) to be

$$\lambda_+ = T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} \qquad \text{and} \qquad \lambda_- = T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}}\,.$$

Clearly, these formulas were obtained earlier in the usual formalism (for details see [4, §5.3 and §6.1] or [7, Section 6.1]). But the above consideration gives a more direct understanding for how these formulas come about.

Another advantage is that it becomes clearer how different contributions to the fermionic projector influence the eigenvalues. We explain this in the example of a left-handed contribution of the form

$$P(x,y) \asymp \chi_L \slashed{u}\,. \tag{2.70}$$

The corresponding contribution to the left-handed component of the closed chain is given by

$$\chi_L A_{xy} \asymp \chi_L \slashed{u}\, P^\varepsilon(y,x)\,.$$

If we substitute $P^\varepsilon(y,x)$ according to (2.67), the factor ι will be contracted in any composite expression either with u or with another factor ι. In both cases, we get contributions of higher order in $\varepsilon/|\vec{\xi}|$. Hence we can disregard the factor ι,

$$\chi_L A_{xy} \asymp -\frac{i}{2}\, \chi_L\, \slashed{u}\, \check{\slashed{\xi}}\, \overline{T^{(-1)}_{[0]}}\,.$$

When multiplying with (2.68), the product with the second summand vanishes. Even more, using the anti-commutation relations, one finds that

$$(A_{xy})^p \, \not{u} \, \check{\not{\xi}} \, (A_{xy})^q = \langle u, \check{\xi} \rangle \left(\frac{1}{2} \not{\xi}^{(-1)}_{[0]} \check{\not{\xi}} \, T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} \right)^{p+q} .$$

This implies that only the eigenvalue λ_{L+} is influenced; more precisely,

$$\lambda_{L+} \asymp -\frac{i}{2} \, u_j \xi^j \, \overline{T^{(-1)}_{[0]}} \qquad \text{and} \qquad \lambda_{L-} \asymp 0 \, .$$

Of course, this result is consistent with earlier computations (see for example [7, Proof of Lemma 7.4 in Appendix B]).

3. The Euler-Lagrange Equations to Degree Five

Before entering the analysis of the EL equations, we briefly recall the basics. Counting with algebraic multiplicities, the closed chain A_{xy} has eight eigenvalues, which we denote by λ^{xy}_{ncs}, where $n \in \{1,2\}$, $c \in \{L,R\}$ and $s \in \{+,-\}$. The corresponding spectral projectors are denoted by F^{xy}_{ncs}. In case of degeneracies, we usually omit the lower indices on which the eigenvalues do not depend. For example, in the case of the four-fold degeneracy $\lambda_{1L+} = \lambda_{2L+} = \lambda_{1R+} = \lambda_{2R+}$, we simply denote the corresponding eigenvalue by λ_+ and the spectral projector onto the four-dimensional eigenspace by F_+.

The considerations in the previous section led us to choosing the Lagrange multiplier $\mu = \frac{1}{8}$ (see (2.17)), and thus a minimizer P is a critical point of the auxiliary action

$$\mathcal{S}[P] = \iint_{M \times M} \mathcal{L}[A_{xy}] \, d^4x \, d^4y$$

with $\mathcal{L}$ according to (1.1),

$$\mathcal{L}[A_{xy}] = \sum_{n,c,s} |\lambda^{xy}_{ncs}|^2 - \frac{1}{8} \left(\sum_{n,c,s} |\lambda^{xy}_{ncs}| \right)^2 = \frac{1}{16} \sum_{n,c,s} \sum_{n',c',s'} \Big(|\lambda^{xy}_{ncs}| - |\lambda^{xy}_{n'c's'}| \Big)^2 .$$

Considering first order variations of P, one gets the EL equations (see [4, §3.5] or for more details [7, eq. (5.20)])

$$[P, Q] = 0 \, , \tag{3.1}$$

where the operator Q has the integral kernel (see [4, §3.5 and §5.4])

$$\begin{aligned} Q(x,y) &= \frac{1}{2} \sum_{ncs} \frac{\partial \mathcal{L}}{\partial \lambda^{xy}_{ncs}} \, F^{xy}_{ncs} \, P(x,y) \\ &= \sum_{n,c,s} \bigg[|\lambda^{xy}_{ncs}| - \frac{1}{8} \sum_{n',c',s'} |\lambda^{xy}_{n'c's'}| \bigg] \, \frac{\overline{\lambda^{xy}_{ncs}}}{|\lambda^{xy}_{ncs}|} \, F^{xy}_{ncs} \, P(x,y) \, . \end{aligned} \tag{3.2}$$

By testing on null lines (see [7, Section 5.2 and Appendix A], one sees that the commutator (3.1) vanishes if and only if Q itself is zero. We thus obtain the EL equations in the continuum limit

$$\boxed{Q(x,y) = 0 \quad \text{if evaluated weakly on the light cone} \, .} \tag{3.3}$$

By relating the spectral decomposition of A_{xy} to that of A_{yx} (see [4, Lemma 3.5.1]), one sees that the operator Q is symmetric, meaning that

$$Q(x,y)^* = Q(y,x) \, . \tag{3.4}$$

As in [7] we shall analyze the EL equations (3.3) degree by degree on the light cone. In this section, we consider the leading degree five, both in the vacuum and in the presence of gauge potentials. In Section 4 we then consider the next degree four.

3.1. **The Vacuum.** Applying the formalism of §2.5 and §2.6 to the ansatz (1.5), (1.6) and (1.9) and taking the partial trace, we obtain according to (2.41) and (2.42) for the vacuum fermionic projector the expression

$$P(x,y) = \frac{i}{2} \begin{pmatrix} 3\,\slashed{\xi} T^{(-1)}_{[0]} + \chi_R\, \tau_{\text{reg}}\, \slashed{\xi} T^{(-1)}_{[R,0]} & 0 \\ 0 & 3\,\slashed{\xi} T^{(-1)}_{[0]} \end{pmatrix} + (\deg < 2)\,, \tag{3.5}$$

where we used a matrix notation in the isospin index. Thus

$$\begin{aligned} \chi_L A_{xy} = \frac{3}{4}\, \chi_L & \begin{pmatrix} 3\,\slashed{\xi} T^{(-1)}_{[0]} \overline{\slashed{\xi} T^{(-1)}_{[0]}} + \tau_{\text{reg}}\, \slashed{\xi} T^{(-1)}_{[0]} \overline{\slashed{\xi} T^{(-1)}_{[R,0]}} & 0 \\ 0 & 3\,\slashed{\xi} T^{(-1)}_{[0]} \overline{\slashed{\xi} T^{(-1)}_{[0]}} \end{pmatrix} \\ & + \slashed{\xi}\,(\deg < 3) + (\deg < 2)\,, \end{aligned}$$

and the right-handed component is obtained by taking the adjoint. The eigenvalues can be computed in the charged and neutrino sectors exactly as in [7, Section 6.1] to obtain

$$\lambda_{2+L} = \lambda_{2+R} = \overline{\lambda_{2-R}} = \overline{\lambda_{2-L}} = 9\, T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} + (\deg < 3) \tag{3.6}$$

and

$$\begin{aligned} \lambda_{1+L} = \overline{\lambda_{1-R}} &= 3\, T^{(0)}_{[0]} \Big(3\, \overline{T^{(-1)}_{[0]}} + \tau_{\text{reg}}\, \overline{T^{(-1)}_{[R,0]}} \Big) + (\deg < 3) \\ \lambda_{1+R} = \overline{\lambda_{1-L}} &= \Big(3\, T^{(0)}_{[0]} + \tau_{\text{reg}}\, T^{(0)}_{[R,0]} \Big)\, 3\, \overline{T^{(-1)}_{[0]}} + (\deg < 3)\,. \end{aligned}$$

The corresponding spectral projectors can be computed exactly as in [4, §5.3 and §6.1] or [7, §6] to

$$F_{1cs} = \begin{pmatrix} \chi_c F_s & 0 \\ 0 & 0 \end{pmatrix}, \qquad F_{2cs} = \begin{pmatrix} 0 & 0 \\ 0 & \chi_c F_s \end{pmatrix}, \tag{3.7}$$

where $F_\pm$ are given by

$$F_\pm := \frac{1}{2} \Big(\mathbb{1} \pm \frac{[\slashed{\xi}, \overline{\slashed{\xi}}]}{z - \overline{z}} \Big) + \slashed{\xi} (\deg \le 0) + (\deg < 0)\,. \tag{3.8}$$

Here the omitted indices of the factors ξ, z and their complex conjugates are to be chosen in accordance with the corresponding factors $T^{(-1)}_\circ$ and $\overline{T^{(-1)}_\circ}$, respectively. In the charged sector, this simply amounts to adding indices ${}^{(-1)}_{[0]}$ to all such factors. In the neutrino sector, however, one must keep in mind the contributions involving τ_{reg}, making it necessary to keep the factors $T^{(n)}_{[R,\circ]}$. More precisely, setting

$$L^{(n)}_{[p]} = T^{(n)}_{[p]} + \frac{1}{3}\, \tau_{\text{reg}}\, T^{(n)}_{[R,p]}\,, \tag{3.9}$$

we obtain

$$2\,\chi_R F_\pm = \mathbb{1} \pm \frac{1}{4L^{(0)}_{[0]} - L^{(-1)}_{[0]}\, \overline{z^{(-1)}_{[0]}}} \left[\not{z}^{(-1)}_{[0]} T^{(-1)}_{[0]} + \frac{1}{3}\, \tau_{\text{reg}}\, \not{z}^{(-1)}_{[R,0]} T^{(-1)}_{[R,0]},\, \overline{\not{z}^{(-1)}_{[0]}} \right]$$
$$2\,\chi_L F_\pm = \mathbb{1} \pm \frac{1}{z^{(-1)}_{[0]}\, \overline{L^{(-1)}_{[0]}} - 4\overline{L^{(0)}_{[0]}}} \left[\not{z}^{(-1)}_{[0]},\, \overline{3\,\not{z}^{(-1)}_{[0]} T^{(-1)}_{[0]} + \tau_{\text{reg}}\, \not{z}^{(-1)}_{[R,0]} T^{(-1)}_{[R,0]}} \right]$$

with the error terms as in (3.7). Moreover, a direct computation shows that (cf. [4, eq. (5.3.23)])

$$F_{nc+}\, P(x,y) = (\deg < 0) \tag{3.10}$$
$$F_{1c-}\, P(x,y) = \chi_c \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P(x,y) + (\deg < 0) \tag{3.11}$$
$$F_{2c-}\, P(x,y) = \chi_c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P(x,y) + (\deg < 0)\,. \tag{3.12}$$

Evaluating the EL equations (3.3) by substituting the above formulas into (3.2), we obtain the three conditions

$$\left(2\left|\overline{T^{(-1)}_{[0]}} T^{(0)}_{[0]}\right| - \left|\overline{L^{(-1)}_{[0]}} T^{(0)}_{[0]}\right| - \left|\overline{T^{(-1)}_{[0]}} L^{(0)}_{[0]}\right| \right) \frac{\overline{T^{(-1)}_{[0]}} T^{(0)}_{[0]}}{\left|\overline{T^{(-1)}_{[0]}} T^{(0)}_{[0]}\right|}\; T^{(-1)}_{[0]} = 0 \tag{3.13}$$
$$\left(3\left|\overline{L^{(-1)}_{[0]}} T^{(0)}_{[0]}\right| - 2\left|\overline{T^{(-1)}_{[0]}} T^{(0)}_{[0]}\right| - \left|\overline{T^{(-1)}_{[0]}} L^{(0)}_{[0]}\right| \right) \frac{\overline{L^{(-1)}_{[0]}} T^{(0)}_{[0]}}{\left|\overline{L^{(-1)}_{[0]}} T^{(0)}_{[0]}\right|}\; L^{(-1)}_{[0]} = 0 \tag{3.14}$$
$$\left(3\left|\overline{T^{(-1)}_{[0]}} L^{(0)}_{[0]}\right| - 2\left|\overline{T^{(-1)}_{[0]}} T^{(0)}_{[0]}\right| - \left|\overline{L^{(-1)}_{[0]}} T^{(0)}_{[0]}\right| \right) \frac{\overline{T^{(-1)}_{[0]}} L^{(0)}_{[0]}}{\left|\overline{T^{(-1)}_{[0]}} L^{(0)}_{[0]}\right|}\; T^{(-1)}_{[0]} = 0\,. \tag{3.15}$$

These three equations must be satisfied in a weak evaluation on the light cone.

To summarize, evaluating the EL equations for the fermionic projector of the vacuum (3.5), we obtain a finite hierarchy of equations to be satisfied in a weak evaluation on the light cone. As the detailed form of these equations is quite lengthy and will not be needed later on, we omit the explicit formulas.

3.2. **The Gauge Phases.** Let us introduce chiral gauge potentials. As the auxiliary fermionic projector (2.41) has seven components, the most general ansatz for chiral potentials would correspond to the local gauge group $\mathrm{U}(7)_L \times \mathrm{U}(7)_R$. However, the causality compatibility conditions (2.49) reduce the local gauge group to

$$\mathrm{U}(6)_L \times \mathrm{U}(6)_R \times \mathrm{U}(1)_R\,, \tag{3.16}$$

where $\mathrm{U}(6)_L$ and $\mathrm{U}(6)_R$ act on the first and last three components, whereas the group $\mathrm{U}(1)_R$ acts on the fourth component. Similar as in [7, Section 6.2], to degree five the gauge potentials describe phase transformations of the left- and right-handed components of the fermionic projector,

$$P^{\text{aux}}(x,y) \to \big(\chi_L\, U_L(x,y) + \chi_R\, U_R(x,y)\big) P^{\text{aux}}(x,y) + (\deg < 2)\,. \tag{3.17}$$

However, as the gauge group (3.16) is non-abelian, the unitary operators $U_{L/R}$ now involve the ordered exponential (for details see [4, §2.5] or [3, Section 2.2])

$$U_{L/R} = \mathrm{Pexp}\Big(-i \int_x^y A^j_{L/R}\, \xi_j \Big) . \tag{3.18}$$

Substituting the gauge potentials corresponding to the gauge group (3.16) and taking the partial trace, we obtain

$$\begin{aligned} \chi_L P(x,y) &= \chi_L\, \frac{i \not{\xi}}{2}\, T^{(-1)}_{[0]} \begin{pmatrix} \hat{U}^{11}_L & \hat{U}^{12}_L \\ \hat{U}^{21}_L & \hat{U}^{22}_L \end{pmatrix} + (\deg < 2) \\ \chi_R P(x,y) &= \chi_R\, \frac{i \not{\xi}}{2} \left[T^{(-1)}_{[0]} \begin{pmatrix} \hat{U}^{11}_R & \hat{U}^{12}_R \\ \hat{U}^{21}_R & \hat{U}^{22}_R \end{pmatrix} + \begin{pmatrix} V\, T^{(-1)}_{[R,0]} & 0 \\ 0 & 0 \end{pmatrix} \right] + (\deg < 2) , \end{aligned} \tag{3.19}$$

where

$$U_{L/R} = \begin{pmatrix} U^{11}_{L/R} & U^{12}_{L/R} \\ U^{21}_{L/R} & U^{22}_{L/R} \end{pmatrix} \in \mathrm{U}(6) , \qquad V \in \mathrm{U}(1) ,$$

and the hat denotes the partial trace,

$$\hat{U}^{ij}_L = \sum_{\alpha,\beta=1}^{3} (U^{ij}_L)^\alpha_\beta , \qquad \hat{U}_R = \sum_{\alpha,\beta=1}^{3} (U_R)^\alpha_\beta . \tag{3.20}$$

At this point it is important to observe that our notation in (3.19) is oversimplified because it does not make manifest that the four matrices $U^{11}_{L/R}$ and $U^{22}_{L/R}$ on the block diagonal describe a mixing of three regularized Dirac seas. Thus when the partial trace is taken, one gets new linear combinations of the regularized Dirac seas, which are then described effectively by the factor $T^{(-1)}_{[0]}$. The analysis in [6] gives a strong indication that an admissible regularization can be obtained only by taking a sum of several Dirac seas and by delicately adjusting their regularizations (more precisely, the property of a distributional $\mathcal{M}P$-product can be arranged only for a sum of at least three Dirac seas). This means that if we take a different linear combination of our three regularized Dirac seas, we cannot expect that the resulting regularization is still admissible. In order to avoid this subtle but important problem, we must impose that each of the four matrices U^{11}_L, U^{22}_L, U^{11}_R and U^{22}_R is a multiple of the identity matrix, because only in this case we get up to a constant the same linear combination of regularized Dirac seas as in the vacuum (for more details and similar considerations see [4, Remark 6.2.3] and [7, §9.3]). This argument shows that the matrices U^{11}_L, U^{22}_L, U^{11}_R and U^{22}_R must be multiples of the identity matrix. The following lemma tells us what these conditions mean for U_L and U_R.

Lemma 3.1. *Suppose that $\mathcal{G} \subset \mathrm{U}(6)$ is a Lie subgroup such that in the standard representation on $\mathbb{C}^6$, every $g \in \mathcal{G}$ is of the form*

$$g = \begin{pmatrix} a \mathbb{1}_{\mathbb{C}^3} & * \\ * & c \mathbb{1}_{\mathbb{C}^3} \end{pmatrix} \qquad \textit{with } a, c \in \mathbb{R}, \tag{3.21}$$

where we used a (3×3) block matrix notation, and the stars stand for arbitrary 3×3-matrices. Then there is a matrix $U \in \mathrm{U}(3)$ such that every $g \in \mathcal{G}$ has the representation

$$g = \begin{pmatrix} a \mathbb{1}_{\mathbb{C}^3} & \overline{b}\, U^* \\ b\, U & c \mathbb{1}_{\mathbb{C}^3} \end{pmatrix} \qquad \textit{with} \qquad \begin{pmatrix} a & \overline{b} \\ b & c \end{pmatrix} \in \mathrm{U}(2) . \tag{3.22}$$

In particular, $\mathcal{G}$ is isomorphic to a Lie subgroup of $\mathrm{U}(2)$.

Proof. For any $A \in T_e\mathcal{G}$ we consider the one-parameter subgroup $V(\tau) = e^{i\tau A}$ $(\tau \in \mathbb{R})$. Evaluating (3.21) to first order in τ, we find that

$$A = \begin{pmatrix} a\mathbb{1}_{\mathbb{C}^3} & Z^* \\ Z & c\mathbb{1}_{\mathbb{C}^3} \end{pmatrix}$$

with a 3×3-matrix Z. Considering (3.21) for the quadratic terms in τ, we find that the matrices ZZ^* and Z^*Z are multiples of the identity matrix. Taking the polar decomposition of Z, we find that

$$A = \begin{pmatrix} a\mathbb{1}_{\mathbb{C}^3} & \overline{b}U^* \\ bU & c\mathbb{1}_{\mathbb{C}^3} \end{pmatrix} \qquad \text{with } a, c \in \mathbb{R} \text{ and } b \in \mathbb{C}\,. \tag{3.23}$$

Exponentiating, one finds that $V(\tau)$ is of the required form (3.22), but with U depending on A.

We next choose two matrices $A, \tilde{A} \in T_e\mathcal{G}$ and represent them in the form (3.23) (where tildes always refer to $\tilde{A}$). It remains that to show that U and $\tilde{U}$ coincide up to a phase,

$$\tilde{U} = e^{i\varphi}\, U \qquad \text{with } \varphi \in \mathbb{R}\,. \tag{3.24}$$

To this end, we consider the one-parameter subgroup $V(\tau) = e^{i\tau(A+\tilde{A})}$. Evaluating (3.21) to second order in τ, we obtain the condition

$$\{A, \tilde{A}\} = \begin{pmatrix} d\mathbb{1}_{\mathbb{C}^3} & * \\ * & e\mathbb{1}_{\mathbb{C}^3} \end{pmatrix} \qquad \text{with } d, e \in \mathbb{R}.$$

Writing out this condition using (3.23), we find that

$$a\tilde{a} + \overline{b}U^*\, \tilde{b}\tilde{U} = d\mathbb{1}_{\mathbb{C}^3}\,. \tag{3.25}$$

Let us show that there is a parameter $\varphi \in \mathbb{R}$ such that (3.24) holds. If b or $\tilde{b}$ vanish, there is nothing to prove. Otherwise, we know from (3.25) that the matrix $U^*\tilde{U}$ is a multiple of the identity matrix. Since this matrix is unitary, it follows that $U^*\tilde{U} = e^{i\varphi}\mathbb{1}_{\mathbb{C}^3}$, proving (3.24). ☐

We point out that the matrix $U \in \mathrm{U}(3)$ is the same for all $g \in \mathcal{G}$; this means that U will be a constant matrix in space-time.

Using the representation (3.22) in (3.19), the left-handed component of the fermionic projector becomes

$$\chi_L P(x,y) = \chi_L\, \frac{i\xi\!\!\!/}{2}\, T^{(-1)}_{[0]} \begin{pmatrix} U_L^{11} & U_L^{12}\, U^*_{\text{MNS}} \\ U_L^{21}\, U_{\text{MNS}} & U_L^{22} \end{pmatrix} + (\deg < 2)\,, \tag{3.26}$$

where $U_L \in U(2)$, and $U_{\text{MNS}} \in \mathrm{U}(3)$ is a constant matrix. The matrix U_{MNS} can be identified with the *MNS matrix* in the electroweak theory. In (3.26), we still need to make sense of the expressions

$$\hat{U}_{\text{MNS}}\, T^{(-1)}_{[0]} \qquad \text{and} \qquad \hat{U}^*_{\text{MNS}}\, T^{(-1)}_{[0]}\,. \tag{3.27}$$

Again, the matrix U_{MNS} describes a mixing of regularized Dirac seas, now even combining the seas with different isospin. Since U_{MNS} is constant, one can take the point of view that we should adjust the regularizations of all six Dirac seas in such a way that the expressions in (3.27) are admissible (in the sense that the fermionic projector has the property of a distributional $\mathcal{M}P$-product; see [6]).

For the right-handed component, the high-energy component $T^{(-1)}_{[R,0]}$ makes the argument a bit more involved. Applying Lemma 3.1 to the right-handed component, we obtain a representation of the form

$$\chi_R P(x,y) = \chi_R \,\frac{i\xi\!\!\!/}{2} \left[T^{(-1)}_{[0]} \begin{pmatrix} U_R^{11} & U_R^{12}\, U^* \\ U_R^{21}\, U & U_R^{22} \end{pmatrix} + \tau_{\mathrm{reg}}\, T^{(-1)}_{[R,0]} \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \right] + (\deg < 2)$$

with $(U_R, V) \in U(2) \times U(1)$ and a fixed matrix $U \in U(3)$. As explained after (3.20), our notation is again a bit too simple in that it does not make manifest that the three Dirac seas and the right-handed high-energy states will in general all be regularized differently, and that only their linear combination is described effectively by the factors $T^{(-1)}_\circ$. With this in mind, we can repeat the argument after (3.20) to conclude that the relative prefactor of the regularization functions in the upper left matrix entry should not be affected by the gauge potentials, i.e.

$$U_R^{11} T^{(-1)}_{[0]} + \tau_{\mathrm{reg}}\, V\, T^{(-1)}_{[R,0]} = \kappa \left(T^{(-1)}_{[0]} + \tau_{\mathrm{reg}}\, T^{(-1)}_{[R,0]} \right) \qquad \text{with } \kappa \in \mathbb{C}\,.$$

In particular, one sees that U_R^{11} must be a phase factor, and this implies that U_R must be a diagonal matrix. Moreover, we find that $V = U_R^{11}$.

Putting our results together, we conclude that the admissible local gauge group is

$$\mathcal{G} = \mathrm{U}(2)_L \times \mathrm{U}(1)_R \times \mathrm{U}(1)_R\,. \tag{3.28}$$

Choosing a corresponding potential $(A_L, A_R^C, A_R^N) \in \mathrm{u}(2) \times \mathrm{u}(1) \times \mathrm{u}(1)$, the interaction is described by the operator

$$\mathcal{B} = \chi_R \begin{pmatrix} A\!\!\!/_L^{11} & A\!\!\!/_L^{12}\, U_{\mathrm{MNS}}^* \\ A\!\!\!/_L^{21}\, U_{\mathrm{MNS}} & A\!\!\!/_L^{22} \end{pmatrix} + \chi_L \begin{pmatrix} A\!\!\!/_R^N & 0 \\ 0 & A\!\!\!/_R^C \end{pmatrix}. \tag{3.29}$$

Thus the U(1)-potentials A_R^N and A_R^C couple to the right-handed component of the two isospin components. The U(2)-potential A_L, on the other hand, acts on the left-handed components, mixing the two isospin components. The U_{MNS}-matrix describe a mixing of the generations in the off-diagonal isospin components of A_L.

In order to analyze the EL equations to degree five in the presence of the above gauge potentials, we need to compute the eigenvalues of the closed chain (see (3.3) and (3.2)). Combining (3.19) with the form of the gauge potentials as specified in (3.28) and (3.29), we obtain

$$\chi_L P(x,y) = \frac{3}{2}\, \chi_L\, i\xi\!\!\!/\, T^{(-1)}_{[0]} \begin{pmatrix} U_L^{11} & \overline{c}\, U_L^{12} \\ c\, U_L^{21} & U_L^{22} \end{pmatrix} + (\deg < 2) \tag{3.30}$$

$$\chi_R P(x,y) = \frac{3}{2}\, \chi_R\, i\xi\!\!\!/ \begin{pmatrix} V_R^N\, L^{(-1)}_{[0]} & 0 \\ 0 & V_R^C\, T^{(-1)}_{[0]} \end{pmatrix} + (\deg < 2) \tag{3.31}$$

with $U_L \in U(2)$ and $V_R^N, V_R^C \in U(1)$, where we again used the notation (3.9) and introduced the complex number

$$c = \frac{1}{3}\, \hat{U}_{\mathrm{MNS}}\,.$$

It follows for the closed chain that

$$\chi_L A_{xy} = \frac{9}{4}\,\chi_L \begin{pmatrix} U_L^{11} & \overline{c}\,U_L^{12} \\ c\,U_L^{21} & U_L^{22} \end{pmatrix} \begin{pmatrix} V_R^N & 0 \\ 0 & V_R^C \end{pmatrix} \begin{pmatrix} \slashed{\xi} T^{(-1)}_{[0]} \overline{\slashed{\xi} L^{(-1)}_{[0]}} & 0 \\ 0 & \slashed{\xi} T^{(-1)}_{[0]} \overline{\slashed{\xi} T^{(-1)}_{[0]}} \end{pmatrix} \tag{3.32}$$
$$+ \slashed{\xi}\,(\deg < 3) + (\deg < 2)\,.$$

When diagonalizing the matrix (3.32), the factor $\overline{L^{(-1)}_{[0]}}$ causes major difficulties because it leads to microscopic oscillations in the eigenvectors. Let us explain this problem in detail. First, it is convenient to use the ι-formalism, because then, similar as explained after (2.68), the contributions $\sim \slashed{\iota}\check{\slashed{\xi}}$ and $\sim \check{\slashed{\xi}}\slashed{\iota}$ act on different subspaces. Thus it remains to diagonalize the 2×2-matrices

$$\begin{pmatrix} U_L^{11} & \overline{c}\,U_L^{12} \\ c\,U_L^{21} & U_L^{22} \end{pmatrix} \begin{pmatrix} V_R^N\, T^{(0)}_{[0]} \overline{L^{(-1)}_{[0]}} & 0 \\ 0 & V_R^C\, T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} \end{pmatrix}$$

and

$$\begin{pmatrix} U_L^{11} & \overline{c}\,U_L^{12} \\ c\,U_L^{21} & U_L^{22} \end{pmatrix} \begin{pmatrix} V_R^N\, T^{(-1)}_{[0]} \overline{L^{(0)}_{[0]}} & 0 \\ 0 & V_R^C\, T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} \end{pmatrix}.$$

The characteristic polynomial involves square roots of linear combinations of the inner matrix elements, describing non-trivial fluctuations of the eigenvalues on the regularization scale ε. Such expressions are ill-defined in the formalism of the continuum limit. A first idea for overcoming this idea would be to extend the formalism such as to include square roots of linear combinations of simple fractions. However, even if one succeeded in extending the continuum limit in this way, it would be unclear how the resulting square root expressions after weak evaluation would depend on the smooth parameters U_L^{ij} and $V_R^{M/N}$. The basic difficulty is that integrating over the microscopic oscillations will in general not preserve the square root structure (as a simple example, an integral of the form $\int_0^\infty \sqrt{a+x}\; f(x)\,dx$ cannot in general be written again as a square root of say the form $\sqrt{b+cx}$). This is the reason why the complications related to the factor $L^{(-1)}_{[0]}$ in (3.32) seem to be of principal nature.

In order to bypass this difficulty, we must restrict attention to a parameter range where the eigenvalues of the above matrices can be computed perturbatively. In order to make the scaling precise, we write τ_{reg} as

$$\tau_{\text{reg}} = (m\varepsilon)^{p_{\text{reg}}} \qquad \text{with} \qquad 0 < p_{\text{reg}} < 2\,. \tag{3.33}$$

Under this assumption, we know that that the relation

$$T^{(n)}_{[p]} = L^{(n)}_{[p]}\, \big(1 + \mathcal{O}\big((m\varepsilon)^{p_{\text{reg}}}\big)\big) \qquad \text{holds pointwise} \tag{3.34}$$

(by "holds pointwise" we mean that if we multiply $T^{(n)}_{[p]} - L^{(n)}_{[p]}$ by any simple fraction and evaluate weakly (2.32), we get zero up to an error of the specified order). Making τ_{reg} small in this sense does not necessarily imply that the above matrices can be diagonalized perturbatively, because we need to compare τ_{reg} to the size of the off-diagonal matrix elements U_R^{12} and U_R^{21}. As they are given as line integrals over the chiral potentials (cf. (3.18)), their size is described by

$$\|A_L^{12}\| \cdot |\vec{\xi}| \qquad \text{and} \qquad \|A_L^{21}\| \cdot |\vec{\xi}|$$

(where $\|.\|$ is a Euclidean norm defined in the same reference frame as $\vec{\xi}$). This leads us to the following two cases:

$$\textbf{(i)} \quad |\vec{\xi}| \gg \frac{(m\varepsilon)^{p_{\text{reg}}}}{\|A_L^{12}\| + \|A_L^{21}\|}\,, \qquad\qquad \textbf{(ii)} \quad |\vec{\xi}| \ll \frac{(m\varepsilon)^{p_{\text{reg}}}}{\|A_L^{12}\| + \|A_L^{21}\|}\,. \tag{3.35}$$

In fact, the computations are tractable in both cases, as we now explain.

Case (i). We expand in powers of τ_{reg}. We begin with the case $\tau_{\text{reg}} = 0$. Then in the vacuum, (3.34) implies that the relations (3.13)–(3.15) are trivially satisfied. If gauge potentials are present, in the above matrices we can factor out the scalar functions $T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}}$ and $T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}}$, respectively. Thus it remains to compute the eigenvalues and spectral projectors of the 2×2-matrix

$$\begin{pmatrix} U_L^{11} & \overline{c}\, U_L^{12} \\ c\, U_L^{21} & U_L^{22} \end{pmatrix} \begin{pmatrix} V_R^N & 0 \\ 0 & V_R^C \end{pmatrix} . \tag{3.36}$$

Lemma 3.2. *The matrix in* (3.36) *is normal (i.e. it commutes with its adjoint). Moreover, its eigenvalues have the same absolute value.*

Proof. We denote the matrix in (3.36) by B and write the two factors in (3.36) in terms of Pauli matrices as

$$B = (a \mathbb{1} + i\vec{v}\vec{\sigma})\, e^{i\varphi} (b \mathbb{1} + i \vec{w}\vec{\sigma})$$

with $a, b, \varphi \in \mathbb{R}$ and $\vec{v}, \vec{w} \in \mathbb{R}^3$. Using the multiplication rules of Pauli matrices, one finds that

$$e^{-i\varphi}\, B = (ab - \vec{v}\vec{w}) \mathbb{1} + i(a\vec{w} + b\vec{v} + \vec{v} \wedge \vec{w})\, \vec{\sigma}\,. \tag{3.37}$$

A short calculation shows that this matrix is normal. Moreover, the eigenvalues of B are computed by

$$e^{i\varphi} \left((ab - \vec{v}\vec{w}) \pm i \left| a\vec{w} + b\vec{v} + \vec{v} \wedge \vec{w} \right| \right) .$$

Obviously, these eigenvalues have the same absolute value. ☐

We denote the eigenvalues and corresponding spectral projectors of the matrix in (3.36) by ν_{nL} and I_n. Then, according to the above lemma,

$$|\nu_{1L}| = |\nu_{2L}| \qquad \text{and} \qquad I_n^* = I_n\,. \tag{3.38}$$

For the left-handed component of the closed chain (3.32) we thus obtain the eigenvalues λ_{nLs} and spectral projectors F_{nLs} given by

$$\lambda_{nLs} = \nu_{nL}\, \lambda_s\,, \qquad F_{nLs} = \chi_L\, I_n\, F_s\,, \tag{3.39}$$

where $\lambda_\pm$ and F_s are given by (cf. (3.6) and (3.8),

$$\lambda_+ = 9\, T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} + (\deg < 3)\,, \qquad \lambda_- = 9\, T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[0]}} + (\deg < 3) \tag{3.40}$$

$$F_\pm = \frac{1}{2} \Big(\mathbb{1} \pm \frac{[\not{\xi}, \overline{\not{\xi}}]}{z - \overline{z}} \Big) + \not{\xi} (\deg \leq 0) + (\deg < 0)\,. \tag{3.41}$$

The spectral decomposition of $\chi_R A_{xy}$ is obtained by complex conjugation,

$$\lambda_{nR\pm} = \nu_{nR}\, \lambda_\pm = \overline{\lambda_{nL\mp}} = \overline{\nu_{nL}}\, \lambda_\pm\,, \qquad F_{nL\pm} = F^*_{nR\mp}\,. \tag{3.42}$$

Combining these relations with (3.38) and (3.40), we conclude that all the eigenvalues of the closed chain have the same absolute value. Thus in view of (3.2), the EL equations are indeed satisfied for $\tau_{\text{reg}} = 0$. In order to treat the higher orders in τ_{reg},

one performs a power expansion up to the required order in the Planck length. The EL equations can be satisfied to every order in τ_{reg} by imposing suitable conditions on the regularization functions. Thus one gets a finite hierarchy of equations to be satisfied in a weak evaluation on the light cone.

Case (ii). We perform a perturbation expansion in the off-diagonal elements U_L^{21} and U_L^{12}. If we set these matrix elements to zero, we again get a spectral representation of the form (3.39)–(3.42), but now with

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \qquad I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.43}$$

and

$$\nu_1 = U_L^{11}\, V_R^N , \qquad \nu_2 = U_R^{22}\, V_R^C .$$

Since the diagonal elements of any $U(2)$-matrix have the same absolute value, it follows that (3.38) again holds. Hence the EL equations are again satisfied in the case $U_L^{21} = 0 = U_L^{12}$. Expanding in powers of U_L^{21} and U_L^{12} again gives a finite hierarchy of equations to be evaluated weakly on the light cone, which can again be satisfied by imposing suitable conditions on the regularization functions.

We conclude that to degree five on the light cone, the EL equations can be satisfied by a suitable choice of the regularization functions, whenever the EL equations have a well-defined continuum limit. Clearly, the detailed computation of admissible regularizations is rather involved. Fortunately, we do not need to work out the details, because they will not be needed later on.

4. The Euler-Lagrange Equations to Degree Four

We now come to the analysis of the EL equations to degree four on the light cone. Before beginning, we clarify our scalings. Recall that the mass expansion increases the upper index of the factors $T^{(n)}_\circ$ and thus decreases the degree on the light cone. In view of the weak evaluation formula (2.32), the mass expansion gives scaling factors $m^2\, \varepsilon|\vec{\xi}|$. Moreover, the parameter τ_{reg} gives scaling factors $(m\varepsilon)^{p_{\text{reg}}}$ (see (3.33)). Unless stated otherwise, we shall only consider the leading order in $(m\varepsilon)^{p_{\text{reg}}}$, meaning that we allow for an error term of the form

$$\Big(1 + \mathcal{O}\big((m\varepsilon)^{p_{\text{reg}}}\big)\Big) . \tag{4.1}$$

Finally, the weak evaluation formulas involve error terms of the form (2.33). Since the contributions to the EL equations of degree four on the light cone involve at least one scaling factor $m^2\, \varepsilon|\vec{\xi}|$ (from the mass expansion) or a factor with the similar scaling $\varepsilon|\vec{\xi}|/\ell_{\text{macro}}^2$ (from the light-cone expansion), the factors $\varepsilon/|\vec{\xi}|$ (which arise from the regularization expansion) give rise to at least one factor $m^2\varepsilon^2$, which can be absorbed into the error term (4.1). Hence, unless stated otherwise, in all the subsequent calculations we neglect the

$$\text{(higher orders in } \varepsilon/\ell_{\text{macro}} \text{ and } (m\varepsilon)^{p_{\text{reg}}}) .$$

For ease in notation, in most computations we omit to write out the corresponding error term $(1 + \mathcal{O}(\varepsilon/\ell_{\text{macro}}) + \mathcal{O}((m\varepsilon)^{p_{\text{reg}}}))$.

4.1. **General Structural Results.** We again denote the eigenvalues of the closed chain A_{xy} by λ^{xy}_{ncs}. These eigenvalues will be obtained by perturbing the eigenvalues with gauge phases as given in (3.39) and (3.42). As a consequence, they will again form complex conjugate pairs, i.e.

$$\lambda^{xy}_{nR\pm} = \overline{\lambda^{xy}_{nL\mp}} \,. \tag{4.2}$$

As the unperturbed eigenvalues all have the same absolute value (see (3.39), (3.38) and (3.40)), to degree four we only need to take into account the perturbation of the square bracket in (3.2). Thus the EL equations reduce to the condition

$$0 = \Delta Q(x,y) := \sum_{n,c,s} \Big[\Delta|\lambda^{xy}_{ncs}| - \frac{1}{8} \sum_{n',c',s'} \Delta|\lambda^{xy}_{n'c's'}| \Big] \, \frac{\overline{\lambda^{xy}_{ncs}}}{|\lambda^{xy}_{ncs}|} \, F^{xy}_{ncs} \, P(x,y) \,, \tag{4.3}$$

where we again evaluate weakly on the light cone and consider the perturbation of the eigenvalues to degree two (also, the superscript xy clarifies the dependence of the eigenvalues on the space-time points).

Here the unperturbed spectral projectors F_{ncs} are given explicitly by (3.39) and (3.41). Moreover, the relations (3.10)–(3.12) can be written in the shorter form

$$F^{xy}_{+} \not{\xi} = (\deg < 0) \,, \qquad F^{xy}_{-} \not{\xi} = \not{\xi} + (\deg < 0) \,. \tag{4.4}$$

Combining these relations with the explicit formulas for the corresponding eigenvalues (see (3.39) and (3.40)) as well as using (4.2), the EL equations (4.3) reduce to the conditions

$$\mathcal{K}_{1L} = \mathcal{K}_{2L} = \mathcal{K}_{1R} = \mathcal{K}_{2R} \qquad \text{mod } (\deg < 4) \,, \tag{4.5}$$

where

$$\mathcal{K}_{nc}(x,y) := \frac{\Delta|\lambda^{xy}_{nc-}|}{|\lambda_-|} \; 3^3 \; T^{(0)}_{[0]} T^{(-1)}_{[0]} \, \overline{T^{(-1)}_{[0]}} \tag{4.6}$$

(for more details see the proof of [7, Lemma 7.1]).

For all the contributions to the fermionic projector of interest in this paper, it will suffice to compute $\Delta|\lambda^{xy}_{nc+}|$ in a perturbation calculation of first or second order. Then the complex numbers $\mathcal{K}_{nc}$ can be recovered as traces of I_n with suitable 2×2-matrices, as the following lemma shows.

Lemma 4.1. *In a perturbation calculation to first order, there are 2×2-matrices $\mathcal{K}_L$ and $\mathcal{K}_R$ such that*

$$\mathcal{K}_{nc} = \text{Tr}_{\mathbb{C}^2} \left(I_n \, \mathcal{K}_c \right) + (\deg < 4) \,. \tag{4.7}$$

In a second order perturbation calculation, one can again arrange (4.7)*, provided that the gauge phases ν_{nc} in the unperturbed eigenvalues* (3.39) *and* (3.42) *must not to be taken into account and that the perturbation vanishes on the degenerate subspaces in the sense that*

$$F_+ \, (\Delta A) \, F_+ = 0 \,. \tag{4.8}$$

Proof. In view of (4.6), it clearly suffices to show that $\Delta|\lambda_{nc+}|$ can be written as such a trace. Writing

$$\Delta|\lambda_{nc+}| = \frac{1}{2|\lambda_+|} \Big((\Delta\lambda_{nc+}) \, \overline{\lambda_+} + \lambda_+ \, \overline{(\Delta\lambda_{nc+})} \Big)$$

and using (4.2), one concludes that it suffices to show that

$$\Delta\lambda_{ncs} = \text{Tr}_{\mathbb{C}^2}(I_n B) \tag{4.9}$$

for a suitable 2×2-matrix $B = B(c,s)$.

The linear perturbation is given by

$$\Delta\lambda_{ncs} = \mathrm{Tr}(F_{ncs}\, \Delta A)\,.$$

As the unperturbed spectral projectors involve a factor I_n (see (3.39) and (3.42)), this is obviously of the form (4.9).

Using (4.8), we have to second order

$$\Delta\lambda_{ncs} = \sum_{n',c'} \frac{1}{\lambda_{ncs} - \lambda_{n'c'(-s)}}\; \mathrm{Tr}(F_{ncs}\, \Delta A\, F_{n'c'(-s)}\, \Delta A)\,. \tag{4.10}$$

Disregarding the gauge phases ν_{cs} in (3.39) and (3.42), we get

$$\begin{aligned} \Delta\lambda_{ncs} &= \sum_{n',c'} \frac{1}{\lambda_s - \lambda_{-s}}\; \mathrm{Tr}(F_{ncs}\, \Delta A\, F_{n'c'(-s)}\, \Delta A) \\ &= \frac{1}{\lambda_s - \lambda_{-s}}\; \mathrm{Tr}(\chi_c\, I_n\, F_s\, \Delta A\, F_{-s}\, \Delta A)\,, \end{aligned}$$

where in the last line we used the form of the spectral projectors in (3.39) and (3.42) and carried out the sums over n' and c'. This is again of the form (4.9). ☐

Instead of analyzing the conditions (4.5), we shall always analyze the stronger conditions

$$\boxed{\mathcal{K}_L(x,y) = \mathcal{K}_R(x,y) = c(\xi)\, \mathbb{1}_{\mathbb{C}^2}\,.} \tag{4.11}$$

This requires a detailed explanation, depending on the two cases in (3.35). In case **(i)**, when the projectors I_n are determined by the chiral gauge potentials, the condition (4.11) can be understood in two different ways. The first, more physical argument is to note that the spectral projectors I_n of the matrix product (3.36) depend on the local gauge potentials A_L and A_R. In order for these potentials to be dynamical, the EL equations should not give algebraic constraints for these potentials (i.e. constraints which involve the potentials but not their derivatives). This can be achieved by demanding that the conditions (4.5) should be satisfied for any choice of the potentials. In view of (4.7), this implies that (4.11) must hold.

To give the alternative, more mathematical argument, let us assume conversely that one of the matrices $\mathcal{K}_L$ or $\mathcal{K}_R$ is *not* a multiple of the identity matrix. Then the perturbation calculation would involve terms mixing the free eigenspaces corresponding to λ_{1cs} and λ_{2cs}. More precisely, to first order one would have to diagonalize the perturbation on the corresponding degenerate subspace. To second order, the resulting contribution to the perturbation calculation would look similar to (4.10), but it would also involve factors of $(\lambda_{1cs} - \lambda_{2cs})^{-1}$. In both cases, the perturbed eigenvalues would no longer be a power series in the bosonic potentials. Analyzing these non-analytic contributions in the EL equations (4.5), one finds that they must all vanish identically. Working out this argument in more detail, one could even derive (4.11) from the EL equations.

In case **(ii)** in (3.35), the projectors I_n are isospin-diagonal (3.43), so that (4.5) only tests the diagonal elements of $\mathcal{K}_c$. Thus at first sight, (4.11) seems a too strong condition. However, even in this case the condition (4.11) can be justified as follows. The left-handed gauge potentials modify the left-handed component of the fermionic projector by generalized phase transformations. If the involved gauge potential is off-diagonal, it makes an off-diagonal components of $P(x,y)$ diagonal and vice versa. As a consequence, satisfying (4.5) in the presence of off-diagonal gauge potentials is

equivalent to satisfying (4.5). We will come back to this argument in more detail in Section 7.

In the next lemma, we express $\Delta Q(x,y)$ in terms of the matrices $\mathcal{K}_c$. This lemma will be needed in Section 6.

Lemma 4.2. *Under the assumptions of Lemma 4.1, the kernel* $\Delta Q(x,y)$ *in* (4.3) *has the representation*

$$\Delta Q(x,y) = \frac{i}{2} \sum_{n,c} \mathrm{Tr}_{\mathbb{C}^2} \big(I_n \, \mathcal{R}_c \big) \, I_n \, \chi_c \, \not{\xi} \;,$$

where

$$\mathcal{R}_L := \mathcal{K}_L - \frac{1}{4} \; \mathrm{Tr}_{\mathbb{C}^2} \big(\mathcal{K}_L + \mathcal{K}_R \big) \, \mathbb{1}_{\mathbb{C}^2} \tag{4.12}$$

(and $\mathcal{R}_R$ is obtained by the obvious replacements $L \leftrightarrow R$).

Proof. Using (4.4) in (4.3), only the summands with $s = -1$ remain. Applying the explicit form of the unperturbed eigenvalues (3.39) and (3.40), the result follows from a straightforward calculation. ☐

The stronger condition (4.11) is then equivalent to demanding that the relations

$$\mathcal{R}_L(x,y) = 0 = \mathcal{R}_R(x,y) \tag{4.13}$$

hold in a weak evaluation on the light cone.

4.2. **The Vacuum.** We begin by analyzing the eigenvalues of the closed chain in the vacuum. As the fermionic projector is diagonal in the isospin index, we can consider the charged sector and the neutrino sector after each other. In the *charged sector*, the eigenvalues can be computed exactly as in [4, §5.3]. Using the notation and conventions in [7], we obtain

$$\begin{aligned}
P(x,y) \;&= \frac{3i}{2} \, \not{\xi} \, T^{(-1)}_{[0]} + \frac{i}{2} \, m^2 \, \acute{Y} \grave{Y} \, T^{(0)}_{[2]} + m \hat{Y} \, T^{(0)}_{[1]} + (\deg < 1) \\
A_{xy} \;&= \frac{3}{4} \, \not{\xi} \overline{\not{\xi}} \left(3 \, T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} + m^2 \, \acute{Y} \grave{Y} \left(T^{(0)}_{[2]} \overline{T^{(-1)}_{[0]}} + T^{(-1)}_{[0]} \overline{T^{(0)}_{[2]}} \right) \right) \\
&\quad + \frac{3i}{2} \, m \hat{Y} \left(\not{\xi} \, T^{(-1)}_{[0]} \overline{T^{(0)}_{[1]}} - T^{(0)}_{[1]} \overline{\not{\xi} \, T^{(-1)}_{[0]}} \right) \\
&\quad + m^2 \hat{Y}^2 \, T^{(0)}_{[1]} \overline{T^{(0)}_{[1]}} + (\deg < 2) \;.
\end{aligned}$$

A straightforward calculation shows that the closed chain has two eigenvalues $\lambda_\pm$, both with multiplicity two. They have the form

$$\begin{aligned}
\lambda_+ &= 9 \, T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} + m^2 \, (\cdots) + (\deg < 2) \\
\lambda_- &= 9 \, T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} + (\cdots) + (\deg < 2) \;,
\end{aligned} \tag{4.14}$$

where $(\cdots)$ stands for additional terms, whose explicit form will not be needed here (for details see [4, eq. (5.3.24)]).

In the *neutrino sector*, by using (2.36) in the ansatz (1.9) and (2.42), after taking the partial trace we obtain

$$\begin{aligned} P(x,y) &= \frac{3i\xi\!\!\!/}{2}\, T^{(-1)}_{[0]} + \chi_R\, \tau_{\text{reg}}\, \frac{i\xi\!\!\!/}{2}\, \big(T^{(-1)}_{[R,0]} + \delta^{-2}\, T^{(0)}_{[R,2]}\big) \\ &\quad + \frac{i}{2}\, \xi\!\!\!/\, m^2\, \acute{Y}\grave{Y}\, T^{(0)}_{[2]} + m\hat{Y}\, T^{(0)}_{[1]} + (\deg < 1) && (4.15) \\ \chi_L A_{xy} &= \frac{3}{4}\, \chi_L\, \xi\!\!\!/\overline{\xi\!\!\!/}\, T^{(-1)}_{[0]} \overline{\big(3\,T^{(-1)}_{[0]} + \tau_{\text{reg}}\, T^{(-1)}_{[R,0]} + \tau_{\text{reg}}\, \delta^{-2}\, T^{(0)}_{[R,2]}\big)} && (4.16) \\ &\quad + \frac{3}{4}\, \xi\!\!\!/\overline{\xi\!\!\!/}\, m^2\, \acute{Y}\grave{Y} \Big(T^{(0)}_{[2]} \overline{T^{(-1)}_{[0]}} + T^{(-1)}_{[0]} \overline{T^{(0)}_{[2]}} \Big) + m^2 \hat{Y}^2\, T^{(0)}_{[1]} \overline{T^{(0)}_{[1]}} && (4.17) \\ &\quad + \frac{3i}{2}\, m\hat{Y} \Big(\xi\!\!\!/\, T^{(-1)}_{[0]} \overline{T^{(0)}_{[1]}} - T^{(0)}_{[1]} \overline{\xi\!\!\!/\, T^{(-1)}_{[0]}} \Big) + (\deg < 2)\,. && (4.18) \end{aligned}$$

The contraction rules (2.37) and (2.39) yield $(\xi\!\!\!/\overline{\xi\!\!\!/})^2 = (z+\overline{z})\, \xi\!\!\!/\overline{\xi\!\!\!/} + z\overline{z}$ and thus

$$(\xi\!\!\!/\overline{\xi\!\!\!/} - z)(\xi\!\!\!/\overline{\xi\!\!\!/} - \overline{z}) = 0\,.$$

This shows that the matrix $\xi\!\!\!/\overline{\xi\!\!\!/}$ has the eigenvalues z and $\overline{z}$. Also applying (3.34), the eigenvalues of the closed chain are computed by

$$\begin{aligned} \lambda_{L+} &= \frac{3}{4}\, z\, T^{(-1)}_{[0]} \overline{\big(3\,L^{(-1)}_{[0]} + \tau_{\text{reg}}\, \delta^{-2}\, T^{(0)}_{[R,2]}\big)} + m^2\,(\cdots) \\ &= 9\, T^{(0)}_{[0]} \overline{L^{(-1)}_{[0]}} + 3\,\tau_{\text{reg}}\, \delta^{-2}\, T^{(0)}_{[0]} \overline{T^{(0)}_{[R,2]}} + m^2\,(\cdots) + (\deg < 2) && (4.19) \\ \lambda_{L-} &= \frac{3}{4}\, T^{(-1)}_{[0]}\, \overline{z\, \big(3\,T^{(-1)}_{[0]} + \tau_{\text{reg}}\, T^{(-1)}_{[R,0]} + \tau_{\text{reg}}\, \delta^{-2}\, T^{(0)}_{[R,2]}\big)} + (\cdots) \\ &= 9\, T^{(-1)}_{[0]} \overline{L^{(0)}_{[0]}} - 3\,\tau_{\text{reg}}\, \delta^{-2}\, T^{(-1)}_{[0]} \overline{T^{(1)}_{\{R,0\}}} + m^2\,(\cdots) + (\deg < 2)\,, && (4.20) \end{aligned}$$

where $L^{(n)}_\circ$ is again given by (3.9), and $m^2\,(\cdots)$ denotes the same contributions as in (4.14) with the masses m_β replaced by the corresponding neutrino masses $\tilde{m}_\beta$. The two other eigenvalues are again obtained by complex conjugation (4.2).

The first summands in (4.19) and (4.20) are of degree three on the light cone and were already analyzed in Section 3. Thus the point of interest here are the summands involving δ. Before analyzing them in detail, we point out that they arise for two different reasons: The term in (4.19) is a consequence of the mass expansion for general surface states. The term in (4.20), on the other hand, corresponds to the last term in the contraction rule (2.39), which takes into account the shear of the surface states.

Let us specify the scaling of the terms involving δ. Recall that the parameter τ_{reg} scales according to (3.33), whereas δ is only specified by (2.35). We want that the general surface and shear states make up for the fact that the masses m_β of the charged fermions are different from the neutrino masses $\tilde{m}_\beta$. Therefore, it would be natural to impose that the summands involving δ should have the same scaling as the contributions $m^2\,(\cdots)$ arising in the standard mass expansion. This gives rise to the scaling

$$\frac{\tau_{\text{reg}}}{\delta^2} \eqsim m^2\,,$$

and thus $\delta \eqsim m\,(m\varepsilon)^{\frac{p_{\text{reg}}}{2}}$. But δ can also be chosen smaller. In this case, the terms involving δ in (4.19) and (4.20) could dominate the contributions by the standard mass expansion. But they do not need to, because their leading contributions may cancel

when evaluated weakly on the light cone. With this in mind, we allow for the scaling

$$\varepsilon \ll \delta \lesssim \frac{1}{m}\,(m\varepsilon)^{\frac{p_{\mathrm{reg}}}{2}}\,. \tag{4.21}$$

Assuming this scaling, by choosing the regularization parameters corresponding to the factors $T^{(0)}_{[R,2]}$ and $T^{(1)}_{\{R,0\}}$ appropriately, we can arrange that (4.3) holds. This procedure works independent of the masses m_β and $\tilde{m}_\beta$.

4.3. **The Current and Mass Terms.** We now come to the analysis of the interaction. More precisely, we want to study the effect of the fermionic wave functions in (2.52) and of the chiral potentials (3.29) in the Dirac operator (2.47) on the EL equations to degree four. As in [7, Section 7] we consider the contribution near the origin in a Taylor expansion around $\xi = 0$.

Definition 4.3. *The integrand in* (2.32) *is said to be of* **order** $o(|\vec{\xi}|^k)$ **at the origin** *if the function η is in the class $o((|\xi^0| + |\vec{\xi}|)^{k+L})$. Likewise, a contribution to the fermionic projector of the form $P(x,y) \asymp \eta(x,y)\, T^{(n)}$ is of the order $o(|\vec{\xi}|^k)$ if $\eta \in o((|\xi^0| + |\vec{\xi}|)^{k+1-n})$.*

Before stating the main result, we define the bosonic current $j_{L/R}$ and the Dirac current $J_{L/R}$ by

$$j^k_{L/R} = \partial^k_{\;j} A^j_{L/R} - \Box A_{L/R} \tag{4.22}$$

$$(J^k_{L/R})^{(i,\alpha)}_{(j,\beta)} = \sum_{l=1}^{n_p} \overline{\Psi^{(j,\beta)}_l}\chi_{R/L}\gamma^k \Psi^{(i,\alpha)}_l - \sum_{l=1}^{n_a} \overline{\Phi^{(j,\beta)}_l}\chi_{R/L}\gamma^k \Phi^{(i,\alpha)}_l\,. \tag{4.23}$$

Note that, due to the dependence on the isospin and generation indices, these currents are 6×6-matrices. We also point out that for the sake of brevity, in (4.22) we omitted the terms quadratic in the potentials which arise for a non-abelian gauge group. But as the form of these quadratic terms is uniquely determined from the well-known behavior under gauge transformations, they could be inserted into all our equations in an obvious way. Similar to the notation (2.54), we denote the partial trace over the generation indices by $\hat{j}$ and $\hat{J}$. Moreover, we introduce the 2×2-matrix-valued vector field $\mathfrak{J}_L$ by

$$\mathfrak{J}^k_L = \hat{J}^k_R\, K_1 + \hat{j}^k_L\, K_2 + \hat{j}^k_R\, K_3 \tag{4.24}$$

$$- 3m^2 \left(\acute{A}^k_L Y \grave{Y} + \acute{Y} Y \grave{A}^k_L \right) K_4 \tag{4.25}$$

$$+ m^2 \left(\hat{A}^k_L\, \acute{Y}\grave{Y} + \acute{Y}\grave{Y}\, \hat{A}^k_L \right) K_4 \tag{4.26}$$

$$- 3m^2 \left(\acute{A}^k_R Y \grave{Y} - 2\acute{Y} A^k_L\, \grave{Y} + \acute{Y} Y \grave{A}^k_R \right) K_5 \tag{4.27}$$

$$- 6m^2 \left(\acute{A}^k_L \grave{Y}\, \hat{Y} + \hat{Y}\, \acute{Y} \grave{A}^k_L \right) K_6 \tag{4.28}$$

$$+ 6m^2 \left(\hat{Y} \acute{A}^k_L \grave{Y} + \acute{Y} \grave{A}^k_L\, \hat{Y} \right) K_7 \tag{4.29}$$

$$+ m^2 \left(\hat{A}^k_L \hat{Y}\hat{Y} + 2\hat{Y}\hat{A}^k_R \hat{Y} + \hat{Y}\hat{Y}\hat{A}^k_L \right) K_6 \tag{4.30}$$

$$- m^2 \left(\hat{A}^k_R \hat{Y}\hat{Y} + 2\hat{Y}\hat{A}^k_L \hat{Y} + \hat{Y}\hat{Y}\hat{A}^k_R \right) K_7\,, \tag{4.31}$$

where $K_1, \ldots, K_7$ are the simple fractions

$$
\begin{aligned}
K_1 &= -\frac{3}{16\pi}\, \frac{1}{\overline{T^{(0)}_{[0]}}} \left[T^{(-1)}_{[0]} T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} - c.c. \right] \\
K_2 &= \frac{3}{4}\, \frac{1}{\overline{T^{(0)}_{[0]}}} \left[T^{(0)}_{[0]} T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]} T^{(0)}_{[0]}} - c.c. \right] \\
K_3 &= \frac{3}{2}\, \frac{1}{\overline{T^{(0)}_{[0]}}} \left[T^{(-1)}_{[0]} T^{(1)}_{[0]}\, \overline{T^{(-1)}_{[0]} T^{(0)}_{[0]}} - c.c. \right] \\
K_4 &= \frac{1}{4}\, \frac{1}{\overline{T^{(0)}_{[0]}}} \left[T^{(0)}_{[0]} T^{(0)}_{[2]}\, \overline{T^{(-1)}_{[0]} T^{(0)}_{[0]}} - c.c. \right] \\
K_5 &= \frac{1}{4}\, \frac{1}{\overline{T^{(0)}_{[0]}}} \left[T^{(-1)}_{[0]} T^{(1)}_{[2]} \overline{T^{(-1)}_{[0]} T^{(0)}_{[0]}} - c.c. \right] \\
K_6 &= \frac{1}{12}\, \frac{T^{(0)}_{[0]} \overline{T^{(0)}_{[0]}}}{\overline{T^{(0)}_{[0]}}}\; \frac{\Big(T^{(0)}_{[1]} \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]} \overline{T^{(0)}_{[1]}} \Big)^2}{T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}}} \\
K_7 &= \frac{1}{12}\, \frac{T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}}}{\overline{T^{(0)}_{[0]}}}\; \frac{\Big(T^{(0)}_{[1]} \overline{T^{(0)}_{[0]}} - T^{(0)}_{[0]} \overline{T^{(0)}_{[1]}} \Big)^2}{T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}}} \;,
\end{aligned}
$$

evaluated weakly on the light cone (2.32) (and *c.c.* denotes the complex conjugate). Similarly, the matrix $\mathfrak{J}_R$ is defined by the replacements $L \leftrightarrow R$.

Lemma 4.4. *The contribution of the bosonic current* (4.22) *and of the Dirac current* (4.23) *to the order* $(\deg < 4) + o(|\vec{\xi}|^{-3})$ *can be written in the form* (4.7) *with*

$$
\mathcal{K}_{L/R} = i\xi_k\, \mathfrak{J}^k_{L/R} \;+\; (\deg < 4) + o\big(|\vec{\xi}|^{-3}\big)\,.
$$

Proof. The perturbation of the eigenvalues is obtained by a perturbation calculation to first and second order (see [4, Appendix G] and [7, Appendix B]). The resulting matrix traces are computed most conveniently in the double null spinor frame $(\mathfrak{f}^{L/R}_{\pm})$ with the methods described in [7, Appendix B]. One finds that ΔA is diagonal on the degenerate subspaces, so that the second order contribution is given by (4.10). Moreover, the gauge phases ν_{nc} in the unperturbed eigenvalues (3.39) and (3.42) only affect the error term $o(|\vec{\xi}|^{-3})$. We conclude that Lemma 4.1 applies, and thus $\mathcal{K}_L$ and $\mathcal{K}_R$ are well-defined.

In order to compute $\mathcal{K}_{L/R}$, we need to take into account the following contributions to the light-cone expansion of the fermionic projector:

$$
\begin{aligned}
\chi_L\, P(x,y) \asymp\; & -\frac{1}{2}\, \chi_L\, \xi\!\!\!/\, \xi_i \int_x^y [0,0\,|\,1]\, j^i_L\, T^{(0)} \\
& - \chi_L \int_x^y [0,2\,|\,0]\, j^i_L\, \gamma_i\, T^{(1)} \\
& - im\, \chi_L\, \xi_i \int_x^y Y A^i_R\, T^{(0)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{im}{2}\, \chi_L \,\slashed{\xi} \int_x^y (Y \slashed{A}_R - \slashed{A}_L Y)\, T^{(0)} \\
&+ im\, \chi_L \int_x^y [0,1\,|\,0] \Big(Y (\partial_j A_R^j) - (\partial_j A_L^j)\, Y \Big)\, T^{(1)} \\
&+ \frac{m^2}{2}\, \chi_L \,\slashed{\xi}\, \xi_i \int_x^y [1,0\,|\,0]\, Y Y A_L^i\, T^{(0)} \\
&+ \frac{m^2}{2}\, \chi_L \,\slashed{\xi}\, \xi_i \int_x^y [0,1\,|\,0]\, A_L^i Y Y\, T^{(0)} \\
&+ m^2\, \chi_L \int_x^y [1,0\,|\,0]\, Y Y \slashed{A}_L\, T^{(1)} \\
&- m^2\, \chi_L \int_x^y [0,0\,|\,0]\, Y \slashed{A}_R\, Y\, T^{(1)} \\
&+ m^2\, \chi_L \int_x^y [0,1\,|\,0]\, \slashed{A}_L\, Y Y\, T^{(1)}
\end{aligned}
$$

(for the derivation see [4, Appendix B] and [3, Appendix A]; cf. also [7, Appendix B]). A long but straightforward calculation (which we carried out with the help of the of the C++ program class_commute and an algorithm implemented in Mathematica) gives the result[2].

We finally mention a rather subtle point in the calculation: According to (3.39) and (3.42), our unperturbed eigenvalues involve gauge phases and can thus be expanded in powers of $A_c^k \xi_k$. As a consequence, we must take into account contributions of the form (4.10) where the factors ΔA involve no gauge potentials, but the unperturbed eigenvalues λ_{ncs} are expanded linearly in $A_c^k \xi_k$. In this case, the corresponding contributions involving no factors of $A_c^k \xi_k$ can be identified with contributions to the eigenvalues in the vacuum in (4.14) and (4.19), (4.20). Using that the vacuum eigenvalues all have the same absolute value, the contributions linear in $A_c^k \xi_k$ can be simplified to obtain the formulas for $\mathfrak{J}_c^k$ listed above. Another, somewhat simpler method to get the same result is to use that the operator Q is symmetric (3.4) (see [4, Lemma 3.5.1]). Thus it suffices to compute the symmetric part $(\Delta Q(x,y) + \Delta Q(y,x)^*)/2$ of the operator ΔQ as defined by (4.3). This again gives the above formulas for $\mathfrak{J}_c^k$, without using any relations between the vacuum eigenvalues. □

Let us briefly discuss the obtained formula for $\mathfrak{J}_R$. The summands in (4.24) involve the chiral gauge currents and Dirac currents; they can be understood in analogy to the current terms in [7, §7.1 and §7.2]. The contributions (4.25)-(4.31) are the mass terms. They are considerably more complicated than in [7, §7.1]. These complications are caused by the fact that we here consider left- and right-handed gauge potentials acting on two sectors, involving a mixing of the generations. In order to clarify the structure of the mass terms, it is instructive to look at the special case of a $U(1)$ vector potential, i.e. $A_L = A_R = A \cdot \mathbb{1}_{\mathbb{C}^2}$ (with a real vector field A). In this case, the terms (4.25) and (4.26) cancel each other (note that $\hat{A}\acute{Y} \cdots = 3\acute{A}Y \cdots$), and (4.27) vanishes. Similarly, the summand (4.28) cancels (4.30), and (4.29) cancels (4.31). Thus the mass terms are zero, in agreement with local gauge invariance.

[2]The C++ program class_commute and its computational output were included as ancillary files to the arXiv submission of [7]. The Mathematica worksheets were included as ancillary files to this arXiv submission.

4.4. **The Microlocal Chiral Transformation.** The simple fractions K_3 and K_5 involve factors $T_\circ^{(1)}$ which have a logarithmic pole on the light cone. Before working out the field equations, we must compensate these logarithmic poles by a suitable transformation of the fermionic projector. We again work with a microlocal chiral transformation as developed in [7, §7.8–§7.11]. As the generalizations to a system of two sectors is not straightforward, we give the necessary constructions step by step. Before beginning, we mention for clarity that in the following sections §4.4 and §4.5 we will construct contributions to $P(x,y)$ which enter the EL equations to degree four only linearly. Therefore, it is obvious that Lemma 4.1 again applies.

As in [7, §7.9] we begin in the homogeneous setting and work in momentum space. Then the logarithmic poles on the light cone correspond to a contribution to the fermionic projector of the form

$$\tilde{P}(k) \asymp \left(\chi_L \, \not{v}_L + \chi_R \, \not{v}_R\right) \delta'(k^2)\, \Theta(-k^0)\,, \tag{4.32}$$

where the vector components v^j_L are Hermitian 2×2-matrices acting on the sector index. In order to generate the desired contribution (4.32), we consider a homogeneous transformation of the fermionic projector of the vacuum of the form

$$\tilde{P}(k) = \acute{U}(k)\, P^{\text{aux}}(k)\, \grave{U}(k)^* \tag{4.33}$$

with a multiplication operator in momentum space $U(k)$. With the operator $U(k)$ we want to modify the states of vacuum Dirac sea with the aim of generating a contribution which can compensate the logarithmic poles. We denote the absolute value of the energy of the states by $\Omega = |k^0|$. We are mainly interested in the regime $m \ll \Omega \ll \varepsilon^{-1}$ where regularization effects play no role. Therefore, we may disregard the right-handed high-energy states and write the vacuum fermionic projector according to (1.5), (1.7) and (1.9). Expanding in the mass, we obtain

$$P^{\text{aux}} = (\not{k} + mY)\, \delta(k^2)\, \Theta(-k^0) - (\not{k} + mY)\, m^2 Y^2\, \delta'(k^2)\, \Theta(-k^0) + (\deg < 0)\,. \tag{4.34}$$

For the transformation $U(k)$ in (4.33) we take the ansatz

$$U(k) = \mathbb{1} + \frac{i}{\sqrt{\Omega}}\, Z(k) \qquad \text{with} \qquad Z = \chi_L\, L^j \gamma_j + \chi_R\, R^j \gamma_j\,, \tag{4.35}$$

where L^j and R^j are 6×6-matrices (not necessarily Hermitian) which act on the generation and sector indices. This ansatz can be regarded as the linear Taylor expansion of the exponential $U = \exp(iZ/\sqrt{\Omega})$, giving agreement to [7, §7.9] (in view of the fact that the quadratic and higher orders of this Taylor expansion dropped out in [7, §7.9], for simplicity we leave them out here). Note that the operator $U(k)$ is in general not unitary (for details see Remark 4.8 below).

Applying the transformation (4.33) and (4.35) to (4.34), only the isospin matrices are influenced. A short calculation gives

$$\begin{aligned}
\chi_L\, \acute{U}\, (\slashed{k} + mY)\, \grave{U}^* &= \chi_L (3\slashed{k} + m\hat{Y}) && (4.36)\\
&\quad + \frac{i}{\sqrt{\Omega}}\, \chi_L \left(\hat{\slashed{L}} \slashed{k} - \slashed{k} \hat{\slashed{R}}^* \right) + \frac{im}{\sqrt{\Omega}}\, \chi_L \left(\acute{\slashed{L}} \grave{Y} - \acute{Y} \grave{\slashed{L}}^* \right) && (4.37)\\
&\quad + \frac{1}{\Omega}\, \chi_L \acute{\slashed{L}} \slashed{k} \grave{\slashed{L}}^* + \frac{m}{\Omega}\, \chi_L \acute{\slashed{L}} Y \grave{\slashed{R}}^* && (4.38)\\
\chi_L\, \acute{U}\, (\slashed{k} + mY)\, m^2 Y^2\, \grave{U}^* &= \chi_L (\slashed{k}\, m^2 \acute{Y} \grave{Y} + m^3\, \acute{Y} Y \grave{Y}) && (4.39)\\
&\quad + \frac{im^2}{\sqrt{\Omega}}\, \chi_L \left(\acute{\slashed{L}} Y \grave{Y}\, \slashed{k} - \slashed{k}\, \acute{Y} Y \grave{\slashed{R}}^* \right) + \frac{im^3}{\sqrt{\Omega}}\, \chi_L \left(\acute{\slashed{L}} Y Y \grave{Y} - \acute{Y} Y Y \grave{\slashed{L}}^* \right) && (4.40)\\
&\quad + \frac{m^2}{\Omega}\, \chi_L \acute{\slashed{L}} \slashed{k}\, Y^2 \grave{\slashed{L}}^* + \frac{m^3}{\Omega}\, \chi_L \acute{\slashed{L}} Y^3 \grave{\slashed{R}}^* && (4.41)
\end{aligned}$$

(and similarly for the right-handed component). Let us discuss the obtained contributions. Clearly, the terms (4.36) and (4.39) are the unperturbed contributions. Generally speaking, due to the factor $\delta(k^2)$ in (4.34), the contributions (4.37) and (4.38) are singular on the light cone and should vanish, whereas the desired logarithmic contribution (4.32) must be contained in (4.40) or (4.41). The terms (4.37) of order $\Omega^{-\frac{1}{2}}$ contribute to the EL equations to degree five on the light cone. Thus in order for them to vanish, we need to impose that

$$\hat{L} = 0 = \hat{R} \tag{4.42}$$

$$\acute{L} \grave{Y} - \acute{Y} \grave{L}^* = 0 = \acute{R} \grave{Y} - \acute{Y} \grave{R}^* \,. \tag{4.43}$$

The last summand in (4.38) does not involve a factor $\slashed{k}$ and is even. As a consequence, it only enters the EL equations in combination with another factor of m, giving rise to a contribution of degree three on the light cone (for details see [7, Lemma B.1]). With this in mind, we may disregard the last summand in (4.38). Similarly, the last summand in (4.41) and the first summand in (4.40) are even and can again be omitted. In order for the second summand in (4.40) to vanish, we demand that

$$\acute{L} Y Y \grave{Y} - \acute{Y} Y Y \grave{L}^* = 0 = \acute{R} Y Y \grave{Y} - \acute{Y} Y Y \grave{R}^* \,. \tag{4.44}$$

Then it remains to consider the first summand in (4.38) and the first summand in (4.41). We thus end up with a chiral contribution to the fermionic projector of the form

$$\chi_L \tilde{P}(k) \asymp \frac{1}{\Omega}\, \chi_L \acute{\slashed{L}} \slashed{k} \grave{\slashed{L}}^*\, \delta(k^2)\, \Theta(-k^0) - \frac{m^2}{\Omega}\, \chi_L \acute{\slashed{L}} \slashed{k}\, Y^2 \grave{\slashed{L}}^*\, \delta'(k^2)\, \Theta(-k^0) \,. \tag{4.45}$$

Note that the conditions (4.42)–(4.44) are linear in L and R, whereas the contribution (4.45) is quadratic.

Before going on, we remark that at first sight, one might want to replace the conditions (4.43) and (4.44) by the weaker conditions

$$\left. \begin{aligned}
\acute{L} \grave{Y} - \acute{Y} \grave{L}^* &= \acute{R} \grave{Y} - \acute{Y} \grave{R}^* = i v_1(k)\, \mathbb{1}_{\mathbb{C}^2} \\
\acute{L} Y Y \grave{Y} - \acute{Y} Y Y \grave{L}^* &= \acute{R} Y Y \grave{Y} - \acute{Y} Y Y \grave{R}^* = i v_3(k)\, \mathbb{1}_{\mathbb{C}^2}
\end{aligned} \right. \tag{4.46}$$

involving two real-valued vector fields v_1 and v_3. Namely, as the resulting contribution to the fermionic projector acts trivially on the isospin index and is symmetric under the replacement $L \leftrightarrow R$, it perturbs the eigenvalues of the closed chain in a way that

the absolute values of all eigenvalues remain equal, so that the EL equations are still satisfied. However, this argument is too simple because the gauge phases must to be taken into account. For the contributions in (4.45), the methods in [7, §7.11] make it possible to arrange that the gauge phases enter in a way which is compatible with the EL equations. For the contributions corresponding to (4.46), however, it is impossible to arrange that the gauge phases drop out of the EL equations. Hence v_1 and v_3 would necessarily enter the EL equations. As the scaling factors $1/\sqrt{\Omega}$ in (4.37) and (4.40) give rise to a different $|\vec{\xi}|$-dependence, these contributions to the EL equations would have a different scaling behavior in the radius. As a consequence, the EL equations would only be satisfied if $v_1 \equiv v_3 \equiv 0$.

For clarity, we want to focus our attention to the component of (4.45) which will give the dominant contribution to the EL equations. For the moment, we only motivate in words how this component is chosen; the detailed justification that the other components can really be neglected will be given in the proof of Proposition 4.6 below. In the EL equations, the chiral component (4.45) is contracted with a factor ξ. This means in momentum space that the main contribution of (4.45) to the EL equations is obtained by contracting with a factor k (this will be justified in detailed in the proof of Proposition 4.6 below). Therefore, we use the anti-commutation relations to rewrite (4.45) as

$$\tilde{P}(k) = \chi_L \, P^j_L(k) \gamma_j + \chi_R \, P^j_R(k) \gamma_j \,.$$

We now contract with k to obtain

$$\begin{aligned} P_L[k] := P^j_L(k)\, k_j &= \frac{1}{\Omega} \left(2 \acute{L}_i \grave{L}^*_j \, k^i k^j - k^2 \, \acute{L}^j \grave{L}^*_j \right) \delta(k^2)\, \Theta(-k^0) \\ &\quad - \frac{m^2}{\Omega} \left(2 \acute{L}_i \, Y^2 \, \grave{L}^*_j k^i k^j - k^2 \, \acute{L}^j \, Y^2 \, \grave{L}^*_j \right) \delta'(k^2)\, \Theta(-k^0) \,. \end{aligned} \tag{4.47}$$

As the factor k^2 vanishes on the mass shell, we may omit the resulting terms (for details see again the proof of Proposition 4.6 below). We thus obtain

$$P_L[k] = \frac{2}{\Omega} \, L[k] \, L[k]^* \, \delta(k^2)\, \Theta(-k^0) - \frac{2}{\Omega} \, L[k] \, m^2 Y^2 \, L[k]^* \, \delta'(k^2)\, \Theta(-k^0) \,, \tag{4.48}$$

where we set $L[k] = \acute{L}_j(k)\, k^j$ (note that $L[k]$ is a 2×6-matrix, and the star simply denotes the adjoint of this matrix). The right-handed component is obtained by the obvious replacements $L \to R$.

Let us work out the conditions needed for generating a contribution of the desired form (4.32). Similar as explained in [7, §7.9], the first summand in (4.48) necessarily gives a contribution to the fermionic projector. For this contribution to drop out of the EL equations, we need to impose that it is vectorial and proportional to the identity matrix, i.e.

$$L[k] \, L[k]^* = R[k] \, R[k]^* = \mathfrak{c}_0(k) \, \mathbb{1}_{\mathbb{C}^2} \tag{4.49}$$

with some constant $\mathfrak{c}_0(k)$. The second summand in (4.48) is of the desired form (4.32). Keeping in mind that we may again allow for a vector contribution proportional to the identity, we get the conditions

$$\begin{aligned} L[k] \, m^2 Y^2 \, L[k]^* &= \frac{\Omega}{2} \, v_L[k] + \mathfrak{c}_2(k) \, \mathbb{1}_{\mathbb{C}^2} \\ R[k] \, m^2 Y^2 \, R[k]^* &= \frac{\Omega}{2} \, v_R[k] + \mathfrak{c}_2(k) \, \mathbb{1}_{\mathbb{C}^2} \,, \end{aligned} \tag{4.50}$$

where we set $v_{L/R}[k] = v^j_{L/R}(k)\, k_j$ (and $\mathfrak{c}_2$ is another free constant). Our task is to solve the quadratic equations (4.49) and (4.50) under the linear constraints (4.42)–(4.44). Moreover, in order to compute the smooth contribution to the fermionic projector, we need to determine the expectation values involving the logarithms of the masses

$$L[k] \left(m^2 Y^2 \log(mY)\right) L[k]^* \qquad \text{and} \qquad R[k] \left(m^2 Y^2 \log(mY)\right) R[k]^* \,. \tag{4.51}$$

We next describe a method for treating the quadratic equations (4.49) and (4.50) under the linear constraints (4.42) (the linear constraints (4.43) and (4.44) will be treated afterwards). We first restrict attention to the left-handed component and consider the corresponding equations in (4.42), (4.49) and (4.50) (the right-handed component can be treated similarly). We write the matrix $L[k]$ in components,

$$L[k] = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} & l_{26} \end{pmatrix} , \tag{4.52}$$

where the matrix entries l_{ab} are complex numbers. We use the linear relations (4.42) to express the third and sixth columns of the matrices by

$$l_{a3} = -l_{a1} - l_{a2} \,, \quad l_{a6} = -l_{a4} - l_{a5} \qquad (a = 1, 2) \,. \tag{4.53}$$

This reduces the number of free parameters to 8 complex parameters, which we combine to the matrix

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \text{with} \qquad \psi_a = (l_{a1}, l_{a2}, l_{a4}, l_{a5}) \,. \tag{4.54}$$

We introduce on $\mathbb{C}^4$ the scalar product $\langle .,. \rangle_0$ as well as the positive semi-definite inner product $\langle .,. \rangle_2$ by

$$\langle \psi_a, \psi_b \rangle_0 = (L[k]\, L[k]^*)^a_b \qquad \text{and} \qquad \langle \psi_a, \psi_b \rangle_2 = (L[k]\, m^2 Y^2\, L[k]^*)^a_b \tag{4.55}$$

(where we implicitly use (4.53) to determine the third and sixth columns of $L[k]$). We represent these scalar products with signature matrices,

$$\langle \psi, \phi \rangle_0 = \langle \psi, S_0\, \phi \rangle_{\mathbb{C}^4} \,, \qquad \langle \psi, \phi \rangle_2 = \langle \psi, S_2\, \phi \rangle_{\mathbb{C}^4} \,.$$

Expressing $\langle .,. \rangle_2$ in terms of $\langle .,. \rangle_0$,

$$\langle \psi, \phi \rangle_2 = \langle \psi, S_0^{-1} S_2\, \phi \rangle_0 \,,$$

the resulting linear operator $S_0^{-1} S_2$ is symmetric with respect to $\langle .,. \rangle_0$. Thus by diagonalizing the matrix $S_0^{-1} S_2$, one can construct an eigenvector basis $e_1, \ldots, e_4$ which is orthonormal with respect to $\langle .,. \rangle_0$, i.e.

$$\langle e_a, e_b \rangle_0 = \delta_{ab} \,, \qquad \langle e_a, e_b \rangle_2 = \mu_a\, \delta_{ab} \,. \tag{4.56}$$

As the matrices have real-valued entries, we can choose the eigenvectors e_a such that all their components are real. Moreover, as the matrices S_0 and S_2 are block-diagonal in the isospin index, we may choose the eigenvectors such that e_1 and e_2 are have isospin up, whereas e_3 and e_4 have isospin down, i.e.

$$e_1, e_2 = (*, *, 0, 0) \,, \qquad e_3, e_4 = (0, 0, *, *)$$

(where the star stands for an arbitrary real-valued entry). Finally, we always order the eigenvalues and eigenvectors such that

$$0 \leq \mu_1 \leq \mu_2 \qquad \text{and} \qquad \mu_3 \leq \mu_4 \,. \tag{4.57}$$

Moreover, Writing the vectors ψ_a in (4.54) in this eigenvector basis,

$$\psi_a = \sum_{d=1}^4 \psi_a^d e_d \,, \tag{4.58}$$

we can express (4.55) in the simpler form

$$(L[k]\,L[k]^*)^a_b = \sum_{d=1}^4 \overline{\psi^d_a} \psi^d_b \,, \qquad (L[k]\,m^2 Y^2\,L[k]^*)^a_b = \sum_{d=1}^4 \mu_d\,\overline{\psi^d_a} \psi^d_b \,. \tag{4.59}$$

Moreover, the linear condition (4.42) is satisfied.

In order to treat the remaining linear constraints (4.43) and (4.44), we decompose the coefficients in (4.58) into their real and imaginary parts,

$$\psi^d_{1/2} = a^d_{1/2} + i\,b^d_{1/2} \,, \qquad (d = 1, \ldots, 4) \,.$$

Considering the diagonal entries of (4.43) and (4.44) shows that

$$b^1_1 = b^2_1 = b^3_2 = b^4_2 = 0 \,.$$

The off-diagonal entries make it possible to express a^3_1, a^4_1 in terms of a^1_2, a^2_2 and b^3_1, b^4_1 in terms of b^1_2, b^2_2, leaving us with the eight real parameters $a^1_1, a^2_1, a^3_2, a^4_2$ and $a^1_2, a^2_2, b^1_2, b^2_2$.

In order to simplify the setting, it is useful to observe that all our constraints are invariant if we multiply the rows of the matrix Ψ in (4.54) by phase factors according to

$$\psi_1 \to e^{i\varphi}\,\psi_1 \,, \qquad \psi_2 \to e^{-i\varphi}\,\psi_2 \qquad \text{with} \qquad \varphi \in \mathbb{R} \,. \tag{4.60}$$

These transformations only affect the off-diagonal isospin components of the left-handed matrix in (4.50). With this in mind, we can assume that this matrix has real components and can thus be decomposed in terms of Pauli matrices as

$$L[k]\,m^2 Y^2 L[k]^* = t\,\mathbb{1} + x\,\sigma^1 + z\,\sigma^3 \,. \tag{4.61}$$

Using this in (4.59) and evaluating the real part of the off-diagonal elements of (4.49), one finds that $b^1_2 = 0 = b^2_2$, leaving us with the six real parameters $a^1_1, a^2_1, a^1_2, a^2_2, a^3_2, a^4_2$. With these six parameters, we need to satisfy three quadratic relations in (4.61) and three quadratic relations in (4.49). This suggests that for given parameters $\mathfrak{c}_0$ and $\mathfrak{c}_2$ as well as t, x, z, there should be a discrete (possibly empty) set of solutions.

In preparation, we analyze the case when all potentials are diagonal in the isospin index.

Example 4.5. (isospin-diagonal potentials) Assume that the parameter x in (4.61) vanishes. Evaluating the real part of the off-diagonal components of (4.49) and (4.61), one finds that $a^1_2 = 0 = a^2_2$. The diagonal components of (4.49) and (4.61) give the quadratic equations

$$(a^1_1)^2 = \frac{-t - z + \mathfrak{c}_0\,\mu_2}{\mu_2 - \mu_1} \,, \qquad\qquad (a^1_2)^2 = \frac{t + z - \mathfrak{c}_0\,\mu_1}{\mu_2 - \mu_1} \tag{4.62}$$

$$(a^2_3)^2 = \frac{-t + z + \mathfrak{c}_0\,\mu_4}{\mu_4 - \mu_3} \,, \qquad\qquad (a^2_4)^2 = \frac{t - z - \mathfrak{c}_0\,\mu_3}{\mu_4 - \mu_3} \,. \tag{4.63}$$

For these equations to admit solutions, we need to assume the non-degeneracies

$$\mu_2 \neq \mu_1 \qquad \text{and} \qquad \mu_3 \neq \mu_4 \,.$$

Then there are solutions if and only if all the squares are non-negative. In view of our sign conventions (4.57), we obtain the conditions

$$\mathfrak{c}_0\,\mu_1 \leq t+z \leq \mathfrak{c}_0\,\mu_2 \qquad \text{and} \qquad \mathfrak{c}_0\,\mu_3 \leq t-z \leq \mathfrak{c}_0\,\mu_4\,. \tag{4.64}$$

Provided that these inequalities hold, the matrix entries a^1_1, a^1_2, a^2_3 and a^2_4 are uniquely determined up to signs. For any solution obtained in this way, one can compute the logarithmic expectation value (4.51).

In order to analyze the conditions (4.64), we first note that changing the constant $\mathfrak{c}_2$ corresponds to adding a constant to the parameter t (see (4.61) and (4.50)). Hence we can always satisfy (4.64) by choosing $\mathfrak{c}_0$ and $\mathfrak{c}_2$ sufficiently large, provided that

$$\mu_1 \leq \mu_4 \qquad \text{and} \qquad \mu_3 \leq \mu_2\,. \tag{4.65}$$

If conversely these conditions are violated, it is impossible to satisfy (4.64) in the case $z = 0$. The physical meaning of the inequalities (4.65) will be discussed in Remark 4.9 below. ◊

In the next proposition, we use a perturbation argument to show that the inequalities (4.65) guarantee the existence of the desired homogeneous transformations even if off-diagonal isospin components are present.

Proposition 4.6. *Assume that the parameters $\mu_1, \ldots, \mu_4$ defined by* (4.56) *and* (4.57) *satisfy the inequalities* (4.65)*. Then for any choice of the chiral potentials v_L and v_R in* (4.32)*, there is a homogeneous chiral transformation of the form* (4.35) *such that the transformed fermionic projector* (4.33) *is of the form*

$$\begin{aligned}
\tilde{P}(k) &= P(k) + (\chi_L \not{v}_L + \chi_R \not{v}_R)\, T^{(1)}_{[3,\mathfrak{c}]} &(4.66)\\
&\quad + \textit{(vectorial)}\; \mathbb{1}_{\mathbb{C}^2}\, \delta(k^2) \Big(1 + \mathcal{O}(\Omega^{-1}) \Big) &(4.67)\\
&\quad + \textit{(vectorial)}\; \mathbb{1}_{\mathbb{C}^2}\, \delta'(k^2) \Big(1 + \mathcal{O}\big(\Omega^{-\frac{1}{2}}\big) \Big) &(4.68)\\
&\quad + \textit{(pseudoscalar or bilinear)}\; \sqrt{\Omega}\, \delta'(k^2) \Big(1 + \mathcal{O}(\Omega^{-1}) \Big) &(4.69)\\
&\quad + \textit{(higher orders in } \varepsilon/|\vec{\xi}|\textit{)}\,. &(4.70)
\end{aligned}$$

Before coming to the proof, we point out that the values of the parameters $\mathfrak{c}_0$ and $\mathfrak{c}_2$ are not determined by this proposition. They can be specified similar as in [7, §7.9] by choosing the homogeneous transformation such that $\mathfrak{c}_0$ is minimal (see also Section 7). In order to clarify the dependence on $\mathfrak{c}_0$ and $\mathfrak{c}_2$, we simply added a subscript $\mathfrak{c}$ to the factor $T^{(1)}_{[3]}$. Similar to [7, eq. (8.3)], this factor can be written in position space as

$$T^{(1)}_{[p,\mathfrak{s}]} = \frac{1}{32\pi^3} \Big(\log|\xi^2| + i\pi\, \Theta(\xi^2)\, \epsilon(\xi^0) \Big) + s_{[p,\mathfrak{s}]}\,,$$

where $s_{[p,\mathfrak{s}]}$ is a real-valued smooth function which depends on the choice of $\mathfrak{c}_0$ and $\mathfrak{c}_2$. In fact, $s_{[p,\mathfrak{s}]}$ may even depend on the isospin components of v_L and v_R; but for ease in notation we shall not make this potential dependence explicit.

Proof of Proposition 4.6. We first show that for sufficiently large $\mathfrak{c}_0$ and $\mathfrak{c}_2$, there are solutions of (4.61) and of the left equation in (4.49). Evaluating the real part of the off-diagonal components of (4.49) and (4.61), we get linear equations in a^1_2 and a^2_2, making it possible to express a^1_2 and a^2_2 in terms of $a^1_1, a^2_1, a^3_2, a^4_2$. These relations do not involve $\mathfrak{c}_0$ nor $\mathfrak{c}_2$. As a consequence, the diagonal components of (4.49) and (4.61)

give a system of equations, which for large parameters $\mathfrak{c}_0$ and $\mathfrak{c}_2$ are a perturbation of the system (4.62) and (4.63). Hence for sufficiently large $\mathfrak{c}_0$ and $\mathfrak{c}_2$, there are solutions by the implicit function theorem.

Repeating the above arguments for the right-handed potentials, we obtain matrices $L[k]$ and $R[k]$ such that (4.49) and (4.50) hold. Moreover, it is clear from our constructions that (4.42), (4.43) and (4.44) are satisfied. It remains to go through all the contributions (4.36)–(4.41) and to verify that they are of the form (4.66)–(4.70). Clearly, (4.36) and (4.39) combine to the summand $P(k)$ in (4.66). The contributions in (4.37) vanish due to (4.42) and (4.43). The second summand in (4.38) as well as the first summand in (4.40) are of the form (4.69). The second summand in (4.40) vanishes in view of (4.44). Hence it really suffices to consider the first summand in (4.38) and the first summand in (4.41), which were combined earlier in (4.45).

It remains to justify the contraction with the momentum k, which led us to analyze (4.48). To this end, we need to consider the derivation of the weak evaluation formulas on the light cone in [4, Chapter 4]. More precisely, the expansion of the vector component in [4, eq. (4.4.6)–(4.4.8)] shows that k and ξ are collinear, up to errors of the order $\varepsilon/|\vec{\xi}|$. Moreover, the terms in (4.47) which involve a factor k^2 are again of the order $\varepsilon/|\vec{\xi}|$ smaller than the terms where the factors k are both contracted to $\acute{L}$ or $\grave{L}^*$. This explains the error term (4.70). □

We remark that the error term (4.70) could probably be improved by analyzing those components of $\acute{L}^j(k)$ which vanish in the contraction $\acute{L}^j(k)\, k^j$. Here we shall not enter this analysis because errors of the order $\varepsilon/|\vec{\xi}|$ appear anyway when evaluating weakly on the light cone (2.32).

Exactly as in [7, §7.10], one can use a quasi-homogeneous ansatz to extend the above methods to a microlocal chiral transformation of the form

$$U(x,y) = \int \frac{d^4k}{(2\pi)^4}\, U\Big(k, v_{L/R}\Big(\frac{x+y}{2}\Big)\Big)\, e^{-ik(x-y)}\,, \tag{4.71}$$

and one introduced the auxiliary fermionic projector is defined via the Dirac equation

$$(U^{-1})^* (i\partial\!\!\!/ - mY)\, U^{-1} \tilde{P}^{\text{aux}} = 0\,. \tag{4.72}$$

This gives the following result.

Proposition 4.7. *Assume that the parameters $\mu_1, \ldots, \mu_4$ defined by* (4.56) *and* (4.57) *satisfy the inequalities* (4.65). *Then for any choice of the chiral potentials v_L and v_R in* (4.32), *there is a microlocal chiral transformation of the form* (4.71) *such that the transformed fermionic projector $\tilde{P} := \acute{U} P^{aux} \grave{U}^*$ is of the form*

$$\begin{aligned}
\tilde{P}(x,y) = P(x,y) &+ (\chi_L \not{v}_L + \chi_R \not{v}_R)\, T^{(1)}_{[3,\mathfrak{c}]}\, \big(1 + \mathcal{O}(|\vec{\xi}|/\ell_{macro})\big) && (4.73)\\
&+ \textit{(vectorial)}\, \mathbb{1}_{\mathbb{C}^2}\, (\deg < 2) + \textit{(pseudoscalar or bilinear)}\, (\deg < 1) && (4.74)\\
&+ \textit{(smooth contributions)} + \textit{(higher orders in } \varepsilon/|\vec{\xi}|\textit{)}\,. && (4.75)
\end{aligned}$$

We conclude this section with two remarks.

Remark 4.8. (Unitarity of U) We now explain why it is would be preferable that the operator U in the microlocal transformation were unitary, and how and to which extent this can be arranged. We begin with the homogeneous setting (4.33) and (4.35). As pointed out after (4.35), the operator U as given by (4.35) is in general not unitary.

However, the following construction makes it possible to replace U by a unitary operator without effecting out results: We first consider the left-handed matrices $L^j(k)$. Note that our analysis only involved the partial trace $\acute{L}[k]$ of these matrices contracted with k. Moreover, by multiplying the columns by a phase (4.60) we could arrange that all the components in (4.51) were real. In this situation, a straightforward analysis shows that there is indeed a Hermitian 6×6-matrix whose partial trace coincides with (4.51). By choosing the other components of $L^j(k)$ appropriately, one can arrange that the matrices $L^j(k)$ are all Hermitian, and (4.51) still holds. Similarly, one can also arrange that the matrices $R^j(k)$ are Hermitian. Replacing the ansatz (4.35) by $U = \exp(iZ/\sqrt{\Omega})$, we get a unitary operator. A straightforward calculation shows that expanding the exponential in a Taylor series, the second and higher orders of this expansion only effect the error terms in Proposition 4.6 (for a similar calculation see [7, §7.9]).

Having arranged that U is unitary has the advantage that the auxiliary fermionic projector defined via the Dirac equation (4.72) is simply given by $\tilde{P}^{\text{aux}} = UPU^*$ (whereas if U were not unitary, the auxiliary fermionic projector would involve unknown smooth correction terms; see the similar discussion for local transformations in [7, §7.7]).

In the microlocal setting (4.71), the transformation U will no longer be unitary, even if the used homogeneous transformations $U(., v_{L/R})$ are unitary for every $v_{L/R}$. Thus it seems unavoidable that the fermionic projector defined via the Dirac equation (4.72) will differ from the operator UPU^* by smooth contributions on the light cone (see also the discussion after [7, eq. (7.86)]). But even then it is of advantage to choose the homogeneous transformations $U(., v_{L/R})$ to be unitary, because then the correction terms obviously vanish in the limit $\ell_{\text{macro}} \to \infty$. More precisely, a straightforward analysis shows that these correction terms are of the order $|\vec{\xi}|/\ell_{\text{macro}}$. $\Diamond$

Remark 4.9. (Lower bound on the largest neutrino mass) The inequalities (4.64) give constraints for the masses of the fermions, as we now explain. Thinking of the interactions of the standard model, we want to be able to treat the case when a left-handed but no right-handed gauge field is present. In this case, $\mathfrak{c}_0$ is nonzero, but the parameter z vanishes for the right-handed component. In view of our sign conventions (4.57), the first inequality in (4.64) implies that $\mathfrak{c}_0 > 0$. Then the inequalities (4.64) yield the necessary conditions (4.65). More precisely, the eigenvalues $\mu_1, \ldots, \mu_4$ are given in terms of the lepton masses by (see also [7, eq. (7.73)])

$$\mu_{1/2} = \frac{1}{3} \left(\tilde{m}_1^2 + \tilde{m}_2^2 + \tilde{m}_3^2 \mp \sqrt{\tilde{m}_1^4 + \tilde{m}_2^4 + \tilde{m}_3^4 - \tilde{m}_1^2\, \tilde{m}_2^2 - \tilde{m}_2^2\, \tilde{m}_3^2 - \tilde{m}_1^2\, \tilde{m}_3^2} \right)$$

$$\mu_{3/4} = \frac{1}{3} \left(m_1^2 + m_2^2 + m_3^2 \mp \sqrt{m_1^4 + m_2^4 + m_3^4 - m_1^2\, m_2^2 - m_2^2\, m_3^2 - m_1^2\, m_3^2} \right).$$

The first inequality in (4.65) is satisfied once the mass m_3 of the τ-lepton is much larger than the neutrino masses, as is the case for present experimental data. However, the second inequality in (4.65) demands that the largest neutrino mass $\tilde{m}_3$ must be at least of the same order of magnitude as m_2. In particular, our model does not allow for a description of the interactions in the standard model if all neutrino masses are too small.

Before comparing this prediction with experiments, one should clearly take into account that we are working here with the naked masses, which differ from the physical

masses by the contributions due to the self-interaction (with a natural ultraviolet cutoff given by the regularization length ε). Moreover, one should consider the possibility of heavy and yet unobserved so-called sterile neutrinos. $\Diamond$

4.5. **The Shear Contributions.** We proceed by analyzing the higher orders in an expansion in the chiral gauge potentials. Qualitatively speaking, these higher order contributions describe generalized phase transformations of the fermionic projector. Our task is to analyze how precisely the gauge phases come up and how they enter the EL equations. The most singular contributions to discuss are the error terms

$$(\text{vectorial})\; \mathbb{1}_{\mathbb{C}^2}\; (\deg = 1) \tag{4.76}$$

in Proposition 4.7. If modified by gauge phases, these error terms give rise to the so-called *shear contributions* by the microlocal chiral transformation. In the setting of one sector, these shear contributions were analyzed in detail in [7, §7.11]. As the adaptation to the present setting of two sectors is not straightforward, we give the construction in detail.

Recall that the gauge phases enter the fermionic projector to degree two according to (3.30) and (3.31). In order to ensure that the error term (4.76) drops out of the EL equations, it must depend on the gauge phases exactly as (3.30), i.e. it must be modified by the gauge phases to

$$\left[\chi_L \begin{pmatrix} U_L^{11} & \overline{c}\, U_L^{12} \\ c\, U_L^{21} & U_L^{22} \end{pmatrix} + \chi_R \begin{pmatrix} V_R^N & 0 \\ 0 & V_R^C \end{pmatrix} \right] \times (\text{vectorial})\; \mathbb{1}_{\mathbb{C}^2}\; (\deg = 1)\,. \tag{4.77}$$

Namely, if (4.77) holds, the corresponding contributions to the closed chain involve the gauge phases exactly as in (3.32), and a straightforward calculation using (3.37) (as well as (3.9) and (3.34)) shows that the eigenvalues of the closed chain all have the same absolute value. If conversely (4.77) is violated, then the eigenvalues of the closed chain are not the same, and the EL equations will be violated (at least without imposing conditions on the regularization functions). We conclude that the transformation law (4.77) is necessary and sufficient for the EL equations to be satisfied to degree five on the light cone.

In order to arrange (4.77), we follow the procedure in [7, §7.11] and write down the Dirac equation for the auxiliary fermionic projector

$$\mathcal{D}\, \tilde{P}^{\text{aux}} = 0\,, \tag{4.78}$$

where $\mathcal{D}$ is obtained from the Dirac operator with chiral gauge fields (see (2.47) and (3.29)) by performing the nonlocal transformation (4.72),

$$\mathcal{D} := (U^{-1})^* \left(i\partial\!\!\!/_x + \chi_L A\!\!\!/_R + \chi_R A\!\!\!/_L - mY \right) U^{-1}\,.$$

For clarity, we begin with the case when U is homogeneous (4.35) (but the gauge potentials $A_{L/R}$ are clearly varying in space-time; the generalization to a microlocal transformation will be carried after (4.82) below). We decompose $\mathcal{D}$ into its even and odd components,

$$\mathcal{D} = \mathcal{D}_{\text{odd}} + \mathcal{D}_{\text{even}}\,,$$

where

$$\mathcal{D}_{\text{odd}} = \chi_L \mathcal{D} \chi_R + \chi_R \mathcal{D} \chi_L \qquad \text{and} \qquad \mathcal{D}_{\text{even}} = \chi_L \mathcal{D} \chi_L + \chi_R \mathcal{D} \chi_R\,.$$

In [7, §7.11] we flipped the chirality of the gauge fields in $\mathcal{D}_{\text{even}}$. As will become clear below, we here need more freedom to modify the gauge potentials in $\mathcal{D}_{\text{even}}$. To this

end, we now simply replace the gauge fields in $\mathcal{D}_{\text{even}}$ by new gauge fields $A^{\text{even}}_{L/R}$ to be determined later,

$$\mathcal{D}^{\text{flip}}_{\text{even}} = \sum_{c=L/R} \chi_c \,(U^{-1})^* \big(i\partial\!\!\!/_x + \chi_L A\!\!\!/^{\text{even}}_R + \chi_R A\!\!\!/^{\text{even}}_L - mY \big) U^{-1}\, \chi_c \,. \tag{4.79}$$

We replace the Dirac equation (4.78) by

$$\big(\mathcal{D}_{\text{odd}} + \mathcal{D}^{\text{flip}}_{\text{even}} \big)\, \tilde{P}^{\text{aux}} = 0 \,.$$

Exactly as in the proof of [7, Proposition 7.12], one sees that the component $\sim \Omega^{-1}$ of P satisfies the Dirac equation involving the chiral gauge potentials $A^{\text{even}}_{L/R}$. In view of (4.45) and (4.49), we see that the left-handed contribution of (4.76) is modified by the chiral gauge potentials to

$$L[k]\ \text{Pexp}\Big(-i \int_x^y (A^{\text{even}}_R)_j\, \xi^j \Big) L[k]^* \,. \tag{4.80}$$

Thus similar as in (3.17), gauge phases appear. The difference is that the chirality is flipped, and moreover here the new potentials $A^{\text{even}}_{L/R}$ enter. A-priori, these potentials can be chosen arbitrarily according to the gauge group (3.16).

The basic difficulty is that the matrix $L[k]$ is non-trivial in the generation index (see (4.52)–(4.54)). Moreover, the gauge potential A_L involves the Maki-Nakagawa-Sakata matrix U_{MNS} (see (3.26)). Therefore, it is not obvious how (4.80) can be related to (3.26). But the following construction shows that for a specific choice of A^{even}_L the connection can be made: We denote the two column vectors of $L[k]^*$ by $\ell_1, \ell_2 \in \mathbb{C}^6$. In view of (4.49), these vectors are orthogonal. We set $\mathfrak{e}_1 = \ell_1/\|\ell_1\|$ and $\mathfrak{e}_4 = \ell_2/\|\ell_2\|$ and extend these two vectors to an orthonormal basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_6$ of $\mathbb{C}^6$. We choose A^{even}_R such that in this basis it has the form

$$A^{\text{even}}_R(k,x) = \begin{pmatrix} A\!\!\!/^{11}_L(x) & A\!\!\!/^{12}_L(x)\, V^* \\ A\!\!\!/^{21}_L(x)\, V & A\!\!\!/^{22}_L(x) \end{pmatrix}, \tag{4.81}$$

where we used a block matrix representation in the subspaces $\langle \mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3 \rangle$ and $\langle \mathfrak{e}_4, \mathfrak{e}_5, \mathfrak{e}_6 \rangle$. Here the potentials $A\!\!\!/^{ij}_L$ are chosen as in (3.29), and $V \in \mathrm{U}(3)$ is an arbitrary unitary matrix. We point out that the whole construction depends on the momentum k of the homogeneous transformation in (4.80), as is made clear by the notation $A^{\text{even}}_R(k,x)$. Substituting the ansatz (4.81) in (4.80) and using that the columns of $L[k]^*$ are multiples of $\mathfrak{e}_1$ and $\mathfrak{e}_4$, we obtain

$$\begin{aligned} &L[k]\, \text{Pexp}\Big(-i \int_x^y (A^{\text{even}}_R)_j\, \xi^j \Big) L[k]^* \\ &\quad = \begin{pmatrix} U^{11}_L & \overline{d}\, U^{12}_L \\ d\, U^{21}_L & U^{22}_L \end{pmatrix} L[k]\, L[k]^* \overset{(4.49)}{=} \begin{pmatrix} U^{11}_L & \overline{d}\, U^{12}_L \\ d\, U^{21}_L & U^{22}_L \end{pmatrix} \mathfrak{c}_0(k)\, \mathbb{1}_{\mathbb{C}^2} \,, \end{aligned}$$

where U^{ij}_L as in (3.30) and $d = \langle \mathfrak{e}_1, V \mathfrak{e}_1 \rangle_{\mathbb{C}^6}$. Choosing V such that d coincides with the parameter c in (3.30), we recover the transformation law of the left-handed component in (4.77). Repeating the above construction for the right-handed component (by flipping the chirality and replacing $L[k]$ by $R[k]$), we obtain precisely the transformation law (4.77).

In order to get into the microlocal setting, it is useful to observe that the k-dependence of A_R^{even} can be described by a unitary transformation,

$$A_R^{\text{even}}(k,x) = W(k)\, A_L(x)\, W(k)^* \qquad \text{with} \qquad W(k) \in \mathrm{U}(6)\,. \tag{4.82}$$

Interpreting W as a multiplication operator in momentum space and A_L as a multiplication operator in position space, we can introduce A_R^{even} as the operator product

$$A_R^{\text{even}} = W A_L W^*\,. \tag{4.83}$$

We point out that the so-defined potential A_R^{even} is non-local. As the microlocal chiral transformation is non-local on the Compton scale, one might expect naively that the same should be true for A_R^{even}. However, A_R^{even} can be arranged to be localized on the much smaller regularization scale ε, as the following argument shows: The k-dependence of W is determined by the matrix entries of $L[k]$. The analysis in §4.4 shows that the matrix entries of $L[k]$ vary in k on the scale of the energy ε^{-1} (in contrast to the matrix Z, which in view of the factor $1/\sqrt{\Omega}$ in (4.35) varies on the scale m). Taking the Fourier transform, the operator W decays in position space on the regularization scale.

This improved scaling has the positive effect that the error term caused by the quasilocal ansatz (4.83) is of the order $\varepsilon/\ell_{\text{macro}}$. Hence the gauge phases enter the left-handed component of the error term (4.76) as

$$\chi_L \begin{pmatrix} U_L^{11} & \overline{c}\, U_L^{12} \\ c\, U_L^{21} & U_L^{22} \end{pmatrix} \big(1 + \mathcal{O}(\varepsilon/\ell_{\text{macro}})\big)(\text{vectorial})\; \mathbb{1}_{\mathbb{C}^2}\; (\deg = 1)\,.$$

Carrying out a similar construction for the right-handed component, we obtain the following result.

Proposition 4.10. *Introducing the potentials $A_{L/R}^{even}$ in the flipped Dirac operator* (4.79) *according to* (4.83)*, the error term* (4.76) *in Proposition 4.7 transforms to*

$$\left[\chi_L \begin{pmatrix} U_L^{11} & \overline{c}\, U_L^{12} \\ c\, U_L^{21} & U_L^{22} \end{pmatrix} + \chi_R \begin{pmatrix} V_R^N & 0 \\ 0 & V_R^C \end{pmatrix} \right] \big(1 + \mathcal{O}(\varepsilon/\ell_{macro})\big)\; \textit{(vectorial)}\; \mathbb{1}_{\mathbb{C}^2}\; (\deg = 1)\,.$$

In this way, we have arranged that the EL equations are satisfied to degree five on the light cone. Note that the above construction involves the freedom in choosing the basis vectors $\mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_5, \mathfrak{e}_6$ as well as the unitary matrix V in (4.81). This will be analyzed in more detail in§5.1.

4.6. **The Energy-Momentum Tensor and the Curvature Terms.** Considering the contribution of the particle and anti-particle wave functions in (2.52) at the origin $x = y$ gives rise to the Dirac current terms as considered in §4.3 (for details see also [7, §7.2]). We now go one order higher in an expansion around the origin $\xi = 0$. Setting $z = (x + y)/2$ and expanding in powers of ξ according to

$$\begin{aligned}
\Psi(x) &= \Psi(z - \xi/2) = \Psi(z) - \frac{1}{2}\, \xi^j \partial_j \Psi(z) + o(|\vec{\xi}|) \\
\Psi(y) &= \Psi(z + \xi/2) = \Psi(z) + \frac{1}{2}\, \xi^j \partial_j \Psi(z) + o(|\vec{\xi}|) \\
\Psi(x)\overline{\Psi(y)} &= \Psi(z)\overline{\Psi(z)} - \frac{1}{2}\, \xi^j \Big(\big(\partial_j \Psi(z)\big)\overline{\Psi(z)} - \Psi(z)\big(\partial_j \overline{\Psi}(z)\big) \Big) + o(|\vec{\xi}|)\,,
\end{aligned}$$

we can write the contribution by the particles and anti-particles as

$$P(x,y) \asymp -\frac{1}{8\pi} \sum_{c=L/R} \chi_c \gamma_k \left(\hat{J}^k_c - i\xi_l \, \hat{T}^{kl}_c \right) + o(|\vec{\xi}|) + \text{(even contributions)} \,,$$

where

$$(T^{kl}_{L/R})^{(i,\alpha)}_{(j,\beta)} = -\operatorname{Im} \sum_{a=1}^{n_p} \overline{\Psi^{(j,\beta)}_a} \chi_{R/L} \gamma^k \partial^l \Psi^{(i,\alpha)}_a + \operatorname{Im} \sum_{b=1}^{n_a} \overline{\Phi^{(j,\beta)}_b} \chi_{R/L} \gamma^k \partial^l \Phi^{(i,\alpha)}_b \,, \qquad (4.84)$$

and similar to (2.54), the hat denotes the partial trace over the generation index. We denote the vectorial component by

$$T^{kl} := T^{kl}_L + T^{kl}_R \,.$$

Taking the trace over the generation and isospin indices, we obtain the *energy-momentum tensor* of the particles and anti-particles.

Lemma 4.11. *The tensors $T^{kl}_{L/R}$,* (4.84)*, give the following contribution to the matrices $\mathcal{K}_{L/R}$ in* (4.7)*,*

$$\mathcal{K}_{L/R} \asymp \hat{T}^{kl}_{R/L} \, \xi_k \, \xi_l \, K_8 \; + \; (\deg < 4) + o\big(|\vec{\xi}|^{-2}\big) \,,$$

where K_8 is the simple fraction

$$K_8 = \frac{3}{16\pi} \, \frac{1}{\overline{\overline{T^{(0)}_{[0]}}}} \left[T^{(-1)}_{[0]} T^{(0)}_{[0]} \, \overline{T^{(-1)}_{[0]}} + c.c. \right] .$$

(note that K_8 differs from K_1 on page 37 in that the term $-c.c.$ has been replaced by $+c.c.$).

The obvious idea for compensating these contributions to the EL equations is to modify the Lorentzian metric. At first sight, one might want to introduce a metric which depends on the isospin index. However, such a dependence cannot occur, as the following argument shows: The singular set of the fermionic projector $P(x,y)$ is given by the pair of points (x,y) with light-like separation. If the metric depended on the isospin components, the singular set would be different in different isospin components. Thus the light cone would “split up” into two separate light cones. As a consequence, the leading singularities of the closed chain could no longer compensate each other in the EL equations, so that the EL equations would be violated to degree five on the light cone.

Strictly speaking, this argument leaves the possibility to introduce a conformal factor which depends on the isospin (because a conformal transformation does not affect the causal structure). However, as the conformal weight enters the closed chain to degree five on the light cone, the EL equations will be satisfied only if the conformal factor is independent of isospin.

The above arguments readily extend to a chiral dependence of the metric: If the left- and right-handed component of the fermionic projector would feel a different metric, then the singular sets of the left- and right-handed components of the closed chain would again be different, thereby violating the EL equations to degree five (for a similar argument for an axial gravitational field see the discussion in [7, §9.3])

Following these considerations, we are led to introducing a Lorentzian metric g_{ij}. Linear perturbations of the metric were studied in [2, Appendix B]. The contributions to the fermionic projector involving the curvature tensor were computed by

$$P(x,y) \asymp \frac{i}{48}\, R_{jk}\, \xi^j \xi^k\, \not{\xi}\, T^{(-1)} \tag{4.85}$$

$$+\frac{i}{24} R_{jk}\, \xi^j \gamma^k\, T^{(0)} + \not{\xi}\,(\deg \leq 1) + (\deg < 1)\,, \tag{4.86}$$

where R_{jk} denotes the Ricci tensor (we only consider the leading contribution in an expansion in powers of $|\vec{\xi}|/\ell_{\text{macro}}$). We refer to (4.85) and (4.86) as the *curvature terms*. More generally, in [11, Appendix A] the singularity structure of the fermionic projector was analyzed on a globally hyperbolic Lorentzian manifold (for details see also [13]). Transforming the formulas in [11, 13] to the the coordinate system and gauge used in [2], one sees that (4.85) and (4.86) also hold non-perturbatively. In particular, the results in [11] show that, to the considered degree on the light cone, no quadratic or even higher order curvature expressions occur. In what follows, we consider (4.85) and (4.86) as a perturbation of the fermionic projector in Minkowski space. This is necessary because at present, the formalism of the continuum limit has only been worked out in Minkowski space. Therefore, strictly speaking, the following results are perturbative. But after extending the formalism of the continuum limit to curved space-time (which seems quite straightforward because the framework of the fermionic projector approach is diffeomorphism invariant), our results would immediately carry over to a globally hyperbolic Lorentzian manifold.

Let us analyze how the curvature terms enter the eigenvalues of the closed chain. We first consider the case when we strengthen (4.21) by assuming that

$$\varepsilon \ll \delta \ll \frac{1}{m}\,(m\varepsilon)^{\frac{p_{\text{reg}}}{2}} \tag{4.87}$$

(the case $\delta \simeq m\,(m\varepsilon)^{\frac{p_{\text{reg}}}{2}}$ will be discussed in Section 8). The assumption (4.87) makes it possible to omit the terms $\sim m^2 R_{ij}$.

Lemma 4.12. *The curvature of the Lorentzian metric gives the following contribution to the matrices $\mathcal{K}_{L/R}$ in* (4.7),

$$\mathcal{K}_L, \mathcal{K}_R \asymp \frac{5}{24}\,\frac{1}{48}\, R_{jk}\, \xi^j \xi^k\; A^0_{xy}\, P_0(x,y) \tag{4.88}$$

$$+\frac{\tau_{reg}}{\delta^2}\, R_{jk}\, \xi^j \xi^k \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} K_{16} \tag{4.89}$$

$$+\, m^2\, R_{jk}\, \xi^j \xi^k\, (\deg = 4) + (\deg < 4) + o\big(|\vec{\xi}|^{-2}\big)\,,$$

where K_{16} is the following simple fraction of degree four,

$$K_{16} = \frac{27}{32}\, \big|T^{(-1)}_{[0]}\big|^2\, \overline{T^{(0)}_{[0]}}^{-1} \left(T^{(1)}_{[R,0]} \overline{L^{(0)}_{[0]}} + L^{(0)}_{[0]} \overline{T^{(1)}_{[R,0]}} \right) \tag{4.90}$$

(and $P_0(x,y)$ and A^0_{xy} denote the vacuum fermionic projector and the closed chain of the vacuum, respectively).

Proof. The contribution (4.85) multiplies the fermionic projector of the vacuum by a scalar factor. Thus it can be combined with the vacuum fermionic projector P_0 to the

expression

$$c_{xy}\, P_0(x,y) \qquad \text{with} \qquad c_{xy} := 1 + \frac{1}{24}\, R_{jk}\, \xi^j \xi^k \,. \tag{4.91}$$

Hence the closed chain and the eigenvalues are simply multiplied by a common prefactor,

$$A_{xy} = c_{xy}^2\, A_{xy}^0 \,, \qquad \lambda_{ncs} = c_{xy}^2\, \lambda_{ncs}^0 \,.$$

As a consequence, the contribution (4.85) can be absorbed in (4.88).

The summand (4.86) is a bit more involved, and we treat it in the ι-formalism. The closed chain is computed by

$$A_{xy} \asymp \frac{3}{16}\, R_{jk}\, \check{\xi}^j \gamma^k\, \check{\xi}\!\!\!/\; T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} \tag{4.92}$$

$$+ \frac{3}{16}\, R_{jk}\, \check{\xi}\!\!\!/\; \check{\xi}^j \gamma^k\, T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[0]}} \,. \tag{4.93}$$

Similar as explained for the chiral contribution after (2.70), the eigenvalues λ_{nc+} are only perturbed by (4.92). More precisely,

$$\lambda_{nc+} \asymp \frac{3}{16}\, R_{jk}\, \xi^j \xi^k\, T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} \,,$$

and the other eigenvalues are obtained by complex conjugation (4.2). In particular, one sees that the eigenvalues are perturbed only by a common prefactor. Combining the perturbation with the eigenvalues of the vacuum, we obtain

$$\lambda_{nc+} = \Big(1 + \frac{1}{48}\, R_{jk}\, \xi^j \xi^k \Big)\, \lambda_{nc+}^0 \,.$$

In view of (4.2), this relation also holds for the eigenvalues λ_{nc-}. We conclude that (4.86) can again be absorbed into (4.88). A short calculation using (4.6) shows that the contributions so far combine precisely to (4.88).

It remains to consider the effects of shear and of the general surface terms. The shear contribution is described by a homogeneous transformation of the spinors which is localized on the scale ε (for details see Appendix A). Since this transformation does not effect the macroscopic prefactor c_{xy} in (4.91), the eigenvalues are again changed only by a common prefactor. Hence (4.85) drops out of the EL equations for the shear states. The contributions (4.92) and (4.93), on the other hand, do not involve ι, and are thus absent for the shear states. We conclude that also (4.86) drops out of the EL equations for the shear states.

We finally consider the general surface states. As (4.92) is a smooth factor times the vacuum fermionic projector, the Ricci tensor again drops out of the EL equations. For the remaining term (4.93), the replacement rule (2.51) yields the contribution of the general mass expansion

$$\chi_R\, P^\varepsilon(x,y) \asymp \frac{i}{24}\, R_{jk}\, \xi^j \gamma^k\, \frac{\tau_{\text{reg}}}{\delta^2} \begin{pmatrix} T^{(1)}_{[R,0]} & 0 \\ 0 & 0 \end{pmatrix} .$$

As a consequence,

$$\begin{aligned}\chi_R\, A_{xy} &\asymp \frac{3}{48}\, R_{jk}\, \xi^j \gamma^k\, \overline{\xi\!\!\!/}\, \frac{\tau_{\mathrm{reg}}}{\delta^2} \begin{pmatrix} T^{(1)}_{[R,0]} \overline{T^{(-1)}_{[0]}} & 0 \\ 0 & 0 \end{pmatrix} \\ \lambda_{nR+} &\asymp \frac{3}{48}\, R_{jk}\, \xi^j \xi^k\, \frac{\tau_{\mathrm{reg}}}{\delta^2}\, T^{(1)}_{[R,0]} \overline{T^{(-1)}_{[0]}}\; \mathrm{Tr}_{\mathbb{C}^2} \left(I_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \\ \mathcal{K}_L &\asymp \frac{27}{32}\, R_{jk}\, \xi^j \xi^k\, \frac{\tau_{\mathrm{reg}}}{\delta^2} \left(T^{(1)}_{[R,0]} \overline{L^{(0)}_{[0]}} + L^{(0)}_{[0]} \overline{T^{(1)}_{[R,0]}} \right) \frac{1}{\overline{T^{(0)}_{[0]}}}\, \big|T^{(-1)}_{[0]}\big|^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .\end{aligned}$$

Similarly,

$$\chi_L\, A_{xy} \asymp \frac{3}{48}\, R_{jk}\, \xi^j \gamma^k\, \overline{\xi\!\!\!/}\, \frac{\tau_{\mathrm{reg}}}{\delta^2} \begin{pmatrix} T^{(-1)}_{[0]} \overline{T^{(1)}_{[R,0]}} & 0 \\ 0 & 0 \end{pmatrix} ,$$

giving the result. □

4.7. **Scalar/Pseudoscalar Potentials, the Higgs Field.** As explained in [7, §8.5], the Higgs potential of the standard model can be identified with suitable scalar/pseudoscalar potentials in the Dirac equation. As shown in [7, Lemma B.1], the contributions by the pseudoscalar potentials to the fermionic projector drop out of the EL equations. The scalar potentials, on the other hand, contribute to the EL equations to degree three on the light cone. As the detailed computations are rather involved, we postpone the analysis of these contributions to a future publication.

5. Structural Contributions to the Euler-Lagrange Equations

In this section, we analyze additional contributions to the EL equations to degree four on the light cone. These contributions will not enter the field equations, but they are nevertheless important because they give constraints for the form of the admissible gauge fields and thus determine the structure of the interaction. For this reason, we call them structural contributions.

5.1. **The Bilinear Logarithmic Terms.** We now return to the logarithmic singularities on the light cone. In §4.3 we computed the corresponding contributions to the EL equations to the order $o(|\vec{\xi}|^{-3})$ at the origin. In §4.4, we succeeded in compensating the logarithmic singularities by a microlocal chiral transformation. The remaining question is how the logarithmic singularities behave in the next order in a Taylor expansion around $\xi = 0$. It turns out that the analysis of this question yields constraints for the form of the admissible gauge fields, as is made precise by the following proposition.

Proposition 5.1. *Assume that the parameter $\mathfrak{c}_2$ in* (4.50) *is sufficiently large and that the chiral potentials in* (3.29) *satisfy the conditions*

$$A^{11}_L - A^N_R = \pm (A^{22}_L - A^C_R) \qquad \textit{for all space-time points} . \tag{5.1}$$

Moreover, in case **(i)** *in* (3.35) *we assume that the MNS matrix and the mass matrix satisfy the relation*

$$\begin{pmatrix} 0 & \acute{U}^*_{MNS} \\ \acute{U}_{MNS} & 0 \end{pmatrix} Y \grave{Y} = \acute{Y} Y \begin{pmatrix} 0 & \grave{U}^*_{MNS} \\ \grave{U}_{MNS} & 0 \end{pmatrix} . \tag{5.2}$$

Then one can arrange by a suitable choice of the basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_6$ and the unitary matrix V in (4.81) *that the contributions to the EL equations $\sim |\vec{\xi}|^{-3} \log |\vec{\xi}|$ vanish.*

If conversely (5.1) *does not hold and if we do not assume any relations between the regularization parameters, then the EL equations of order* $|\vec{\xi}|^{-3} \log |\vec{\xi}|$ *are necessarily violated at some space-time point.*

The importance of this proposition is that it poses a further constraint on the form of the chiral gauge potentials.

The remainder of this section is devoted to the proof of Proposition 5.1. Generally speaking, our task is to analyze how the gauge phases enter the logarithmic singularities of the fermionic projector. We begin with the logarithmic current term

$$\chi_L P^{\text{aux}}(x,y) \asymp -2\, \chi_L \int_x^y [0,0\,|\,1]\, j_L^i\, \gamma_i\, T^{(1)}\,,$$

which gives rise to the last summand in (4.24) (and similarly for the right-handed component; for details see [7, eq. (B.17)–(B.18)] or [4, Appendix B]). According to the general rules for inserting ordered exponentials (see [3, Definition 2.9] or [4, Definition 2.5.5]), the gauge potentials enter the logarithmic current term according to

$$-2\, \chi_L \int_x^y [0,0\,|\,1]\; \mathrm{Pe}^{-i \int_x^z A_L^k (z-x)_k}\, j_L^i(z)\, \gamma_i\, \mathrm{Pe}^{-i \int_z^y A_L^l (y-z)_l}\; T^{(1)} \tag{5.3}$$

(where $\mathrm{Pe} \equiv \mathrm{Pexp}$ again denotes the ordered exponential (3.18)). Performing a Taylor expansion around $\xi = 0$ gives

$$\chi_L P^{\text{aux}}(x,y) \asymp -\frac{1}{3}\, \chi_L\, j_L^i\Big(\frac{x+y}{2}\Big) \gamma_i\, T^{(1)} \tag{5.4}$$

$$+\frac{i}{6}\, \chi_L \left(A_L^j \xi_j\; j_L^i \gamma_i + j_L^i \gamma_i\; A_L^j \xi_j \right)\, T^{(1)} + o(|\vec{\xi}|) \tag{5.5}$$

(note that in (5.5) it plays no role if the functions are evaluated at x or y because the difference can be combined with the error term).

We arranged by the microlocal chiral transformation that the logarithmic singularity of (5.4) is compensated by the second summand in (4.73). Since both (5.4) and (4.71) involve the argument $(x+y)/2$, the logarithmic singularities compensate each other even if x and y are far apart (up to the error terms as specified in (5.5) and Proposition 4.7). Thus it suffices to analyze how the gauge phases enter (4.73). To this end, we adapt the method introduced after (4.80) to the matrix products in (4.50). Beginning with the left-handed component, the square of the mass matrix is modified by the gauge phases similar to (5.3) and (5.4), (5.5) by

$$\begin{aligned} \chi_L\, m^2 Y^2 &\to \chi_L \int_x^y \mathrm{Pe}^{-i \int_x^z A_L^k (z-x)_k}\, m^2 Y^2\, \mathrm{Pe}^{-i \int_z^y A_L^l (y-z)_l}\; dz \\ &= \chi_L\, m^2 Y^2 - \frac{i}{2}\, \chi_L\, m^2 \left(A_L^j \xi_j\, Y^2 + Y^2\, A_L^j \xi_j \right) + o(|\vec{\xi}|) \end{aligned}$$

(for details see [3, Section 2 and Appendix A]). When using this transformation law in (4.50), we need to take into account that, similar to (4.80), the chiral gauge potentials $A_{L/R}$ must be replaced by $A_{R/L}^{\text{even}}$. Thus we need to compute the expectation values

$$\chi_L P(x,y) \asymp -\frac{i}{2}\, m^2\, L[k] \Big(A_R^{\text{even}}[\xi]\, Y^2 + Y^2\, A_R^{\text{even}}[\xi] \Big) L[k]^*\,, \tag{5.6}$$

where the square bracket again denotes a contraction, $A_R^{\text{even}}[\xi] \equiv (A_R^{\text{even}})_k\, \xi^k$.

Again choosing the basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_6$, the potential A_R^{even} is of the form (4.81). Now we must treat the diagonal and the off-diagonal elements of A_L separately. Obviously,

the diagonal entries in (4.81) map the eigenvectors $\mathfrak{e}_1$ and $\mathfrak{e}_4$ to each other. Hence (5.6) gives rise to the anti-commutator

$$\begin{aligned}\chi_L P(x,y) &\asymp -\frac{i}{2} \left\{ \begin{pmatrix} A_L^{11}[\xi] & 0 \\ 0 & A_L^{22}[\xi] \end{pmatrix}, L[k]\, m^2 Y^2 L[k]^* \right\} \\ &\overset{(4.50)}{=} -\frac{i}{2} \left\{ \begin{pmatrix} A_L^{11}[\xi] & 0 \\ 0 & A_L^{22}[\xi] \end{pmatrix}, \frac{\Omega}{2}\, v_L[k] + \mathfrak{c}_2(k)\, \mathbb{1}_{\mathbb{C}^2} \right\}. \end{aligned} \tag{5.7}$$

For the off-diagonal elements of A_L, the matrices V and V^* in (4.81) make the situation more complicated. For example, the lower left matrix entry in (4.81) maps $\mathfrak{e}_1$ to a non-trivial linear combination of $\mathfrak{e}_4, \mathfrak{e}_5, \mathfrak{e}_6$, i.e. for any 6×6-matrix B,

$$L[k]\, B \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} L[k]^* = \begin{pmatrix} \|\ell_1\|^2\, (B^1_4 V^1_1 + B^1_5 V^2_1 + B^1_6 V^3_1) & 0 \\ \|\ell_2\|\, \|\ell_1\|\, (B^4_4 V^1_1 + B^4_5 V^2_1 + B^4_6 V^3_1) & 0 \end{pmatrix}.$$

Similarly, the upper right matrix entry in (4.81) maps $\mathfrak{e}_4$ to a non-trivial linear combination of $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3$. As a consequence, the off-diagonal elements of A_L yield a contribution to (5.6) of the general form

$$\chi_L P(x,y) \asymp A_L^{12}[\xi]\, G(k) + A_L^{21}[\xi]\, G(k)^* \qquad \text{with} \qquad G = \begin{pmatrix} G^{11} & 0 \\ G^{12} & G^{22} \end{pmatrix}. \tag{5.8}$$

Here the 2×2-matrix $G(k)$ depends on $\mathfrak{c}_2$ and $v_L[k]$ as well as on the choice of the basis vectors $\mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_5, \mathfrak{e}_6$ and the matrix V in (4.81). Counting the number of free parameters, one sees that $G(k)$ can be chosen arbitrarily, up to inequality constraints which come from the fact that V must be unitary and that the entries of the matrix $m^2 Y^2$ in the basis $(\mathfrak{e}_1, \ldots, \mathfrak{e}_6)$ cannot be too large due to the Schwarz inequality. These inequality constraints can always be satisfied by suitably increasing the parameter $\mathfrak{c}_2$. For this reason, we can treat $G(k)$ as an arbitrary matrix involving three free real parameters.

The right-handed component of the microlocal chiral transformation can be treated similarly. The only difference is that the right-handed gauge potentials are already diagonal in view of (3.29). Thus we obtain in analogy to (5.7)

$$\chi_R P(x,y) \asymp -\frac{i}{2} \left\{ \begin{pmatrix} A_R^N[\xi] & 0 \\ 0 & A_R^C[\xi] \end{pmatrix}, \frac{\Omega}{2}\, v_R[k] + \mathfrak{c}_2(k)\, \mathbb{1}_{\mathbb{C}^2} \right\}, \tag{5.9}$$

whereas (5.7) has no correspondence in the right-handed component.

Comparing (5.3) with (5.7), (5.8) and (5.9), one sees that the transformation laws are the same for the diagonal elements of A_L and A_R. For the off-diagonal elements of A_L, we can always choose $G(k)$ such as to get agreement with (5.3). We conclude that by a suitable choice of $G(k)$ we can arrange that the transformation law (5.3) agrees with (5.7), (5.8) and (5.9). As a consequence, the logarithmic poles of the current terms are compensated by the microlocal chiral transformation, even taking into account the gauge phases to the order $o(|\xi|)$.

We next consider the logarithmic mass terms

$$\chi_L P^{\text{aux}}(x,y) \asymp m^2\, \chi_L \int_x^y [1,0\,|\,0]\, YY \not{A}_L\, T^{(1)} \tag{5.10}$$

$$- m^2\, \chi_L \int_x^y [0,0\,|\,0]\, Y \not{A}_R\, Y\, T^{(1)} \tag{5.11}$$

$$+ m^2\, \chi_L \int_x^y [0,1\,|\,0]\, \not{A}_L\, YY\, T^{(1)}\,, \tag{5.12}$$

which give rise to the summand (4.27) (for details see [7, eq. (B.29)–(B.31)] or [4, Appendix B]). Here the gauge phases enter somewhat differently, as we now describe.

Lemma 5.2. *Contracting the logarithmic mass terms* (5.10)–(5.12) *with a factor* ξ *and including the gauge phases, we obtain*

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{Tr}\left(i\xi\!\!\!/\,\chi_L\,P^{\mathrm{aux}}(x,y)\right) \\
&\quad\asymp \frac{im^2}{2}\left(A^j_L(z_1)\,YY - 2Y A^j_R(z_2)\,Y + YY A^j_L(z_3)\right)\xi_j\,T^{(1)} && (5.13)\\
&\quad\quad - \frac{m^2}{8}\Big((A^j_L\xi_j)(A^k_L\xi_k)YY + 2(A^j_L\xi_j)YY(A^k_L\xi_k) + YY(A^j_L\xi_j)(A^k_L\xi_k)\Big)\,T^{(1)} && (5.14)\\
&\quad\quad + \frac{m^2}{2}\,Y(A^j_R\xi_j)(A^k_R\xi_k)\,Y\,T^{(1)} + o\big(|\vec{\xi}|^2\big)\,, && (5.15)
\end{aligned}$$

where

$$z_1 = \frac{3x+y}{4}\,, \qquad z_2 = \frac{x+y}{2}\,, \qquad z_3 = \frac{x+3y}{4}\,.$$

The right-handed component is obtained by the obvious replacements $L \leftrightarrow R$.

Proof. Following the method of [3, proof of Theorem 2.10], we first choose a special gauge and then use the behavior of the fermionic projector under chiral gauge transformations. More precisely, by a chiral gauge transformation we can arrange that A_L and A_R vanish identically along the line segment $\overline{xy}$. In the new gauge, the mass matrix Y is no longer constant, but it is to be replaced by dynamical mass matrices $Y_{L/R}(x)$ (see [3, eq. (2.8)] or [4, eq. (2.5.9)]). Performing the light-cone expansion in this gauge, a straightforward calculation yields

$$\frac{1}{2}\,\mathrm{Tr}\left(i\xi\!\!\!/\,\chi_L\,P^{\mathrm{aux}}(x,y)\right) \asymp m^2\Big(Y_L(x)\,Y_R(y) - \int_x^y (Y_L\,Y_R)(z)\,dz\Big)\,T^{(1)}\,.$$

Transforming back to the original gauge amounts to inserting ordered exponentials according to the rules in [3, Definition 2.9] or [4, Definition 2.5.5]. We thus obtain

$$\begin{aligned}
\frac{1}{2}\,\mathrm{Tr}\left(i\xi\!\!\!/\,\chi_L\,P^{\mathrm{aux}}(x,y)\right) &\asymp m^2\,Y\,\mathrm{Pe}^{-i\int_x^y A^j_R \xi_j}\,Y\,T^{(1)} \\
&\quad - m^2 \int_x^y \mathrm{Pe}^{-i\int_x^z A^j_L\,(z-x)_j}\;YY\;\mathrm{Pe}^{-i\int_z^y A^j_L\,(y-z)_j}\;dz\,T^{(1)}\,.
\end{aligned} \tag{5.16}$$

Expanding in powers of ξ gives the result. $\square$

The contribution (5.13) is the mass term which we already encountered in (4.27). In contrast to (5.4) and (4.71), the term (5.13) does not only depend on the variable $(x+y)/2$. However, taking the partial trace, for a *diagonal* potential A_L we obtain in view of (3.29) that

$$\begin{aligned}
\Big(\acute{A}^j_L(z_1)\,Y\grave{Y} + \acute{Y}Y\,\grave{A}^j_L(z_2)\Big)\,\xi_j &= \Big(A^j_L(z_1) + A^j_L(z_2)\Big)\,\xi_j\,\acute{Y}\grave{Y} \\
&= 2A^j_L\Big(\frac{x+y}{2}\Big)\,\xi_j\,\acute{Y}\grave{Y} + o(|\vec{\xi}|)\,.
\end{aligned} \tag{5.17}$$

This makes it possible to compensate the logarithmic singularity of (5.13) by the second term in (4.73), up to the specified error terms. For the *off-diagonal potentials*, the situation is more complicated and depends on the two cases in (3.35). If we are in case **(i)** and (5.2) is satisfied, then (5.17) also holds for the off-diagonal potentials. As a consequence, the logarithmic singularity of (5.13) can again be compensated

by the second term in (4.73). However, if we do not impose (5.2), then it seems impossible to compensate the off-diagonal logarithmic terms by a microlocal chiral transformation (4.73). If we assume instead that we are in case **(ii)** in (3.35), then the spectral projectors I_n are diagonal (3.43), so that off-diagonal potentials are irrelevant as they do not enter the EL equations (4.5). We conclude that we can compensate the logarithmic singularities of (5.13) in case **(i)** under the additional assumptions (5.2), and in case **(ii)** without any additional assumptions.

The terms (5.14) and (5.15), on the other hand, are quadratic in the chiral gauge potentials. Analyzing whether these terms are compatible with the transformation law (5.7), (5.8) and (5.9) of the microlocal chiral transformation gives the following result.

Lemma 5.3. *Consider the component of the fermionic projector which involves a bilinear tensor field and has a logarithmic pole on the light cone,*

$$P(x,y) \asymp \Big(\chi_L h^{ij}_L(x,y) \gamma_i \xi_j + \chi_R h^{ij}_R(x,y) \gamma_i \xi_j \Big)\, T^{(1)} \tag{5.18}$$

(where $h^{ij}_{L/R}$ is a smooth tensor field acting as a 2×2-matrix on the isospin index). If (5.1) *holds and $\mathfrak{c}_2$ is sufficiently large, then one can arrange by a suitable choice of the basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_6$ and of the unitary matrix V in* (4.81) *that*

$$\chi_L\, h^{ij}_L + \chi_R\, h^{ij}_R = h^{ij}\, \mathbb{1}_{\mathbb{C}^2} \tag{5.19}$$

(where h^{ij} is a suitable tensor field which acts trivially on the isospin index). If conversely (5.1) *does not hold, then* (5.19) *is necessarily violated at some space-time point.*

Proof. We first analyze the right-handed component. If (5.13) is transformed according to (5.9), we could argue just as for the logarithmic current terms to conclude that the contribution of the form (5.18) vanishes. Therefore, it suffices to consider the terms obtained by subtracting from (5.13)–(5.15) the term (5.13) transformed according to (5.9). This gives the second order terms

$$\begin{aligned} B := \;& \frac{m^2}{8} \Big(A_R[\xi]^2\, YY + 2 A_R[\xi]\, YY A_R[\xi] + YY A_R[\xi]^2 \Big)\, T^{(1)} \\ & - \frac{m^2}{2} \Big(A_R[\xi]\, Y A_L[\xi]\, Y - Y A_L[\xi]^2\, Y + Y A_L[\xi]\, Y A_R[\xi] \Big)\, T^{(1)} \,. \end{aligned}$$

We decompose A_L into its diagonal and off-diagonal elements, denoted by

$$A_L[\xi] = A^d_L[\xi] + A^o_L[\xi] \,.$$

A straightforward calculation using the identity

$$A_L[\xi]^2 = A^d_L[\xi]^2 + A^o_L[\xi]^2 + \{ A^d_L[\xi], A^o_L[\xi] \}$$

gives

$$B = \frac{m^2}{2}\, Y \left(\big(A_R[\xi] - A^d_L[\xi] \big)^2 + A^o_L[\xi]^2 \right) Y\, T^{(1)} \tag{5.20}$$

$$\quad - \frac{m^2}{2}\, Y \left\{ A_R[\xi] - A^d_L[\xi], A^o_L[\xi] \right\} Y\, T^{(1)} \,. \tag{5.21}$$

Clearly, the matrix $A^o_L[\xi]^2$ is a multiple of the identity matrix. The matrix $(A_R[\xi] - A^d_L)^2$, on the other hand, is a multiple of the identity matrix if and only if (5.1) holds. The anti-commutator in (5.21) vanishes on the diagonal. It vanishes provided

that (5.1) holds. We conclude that the contribution (5.20) and (5.21) acts trivially on the isospin index if and only if (5.1) holds. In this case,

$$B = \frac{m^2}{2}\, Y \left(\big(A_R[\xi] - A_L^d[\xi]\big)^2 + A_L^o[\xi]^2 \right) Y\, T^{(1)} \,. \tag{5.22}$$

It remains to show that under the assumption (5.1), we can arrange that the corresponding left-handed contribution, which we again denote by B, is also a multiple of the identity matrix, and that it coincides with (5.22). In order to compute the relevant left-handed contribution, we subtract from (5.14) and (5.15) the term (5.13) transformed according to (5.7),

$$\begin{aligned} B = \;& \frac{m^2}{4} \Big\{ A_L^e[\xi], \big(A_L[\xi]\, YY - 2Y A_R[\xi]\, Y + YY A_L[\xi]\big) \Big\}\, T^{(1)} \\ & - \frac{m^2}{8} \left(A_L[\xi]^2\, YY + 2A_L[\xi]\, YY A_L[\xi] + YY A_L[\xi]^2 \right) T^{(1)} + \frac{m^2}{2}\, Y A_R[\xi]^2\, Y\, T^{(1)} \,. \end{aligned}$$

In view of (5.8), we can add contributions which involve A_L^{12} or A_L^{21}. A short calculation shows that in this way, we can arrange that B is again of the form (5.22). □

The next lemma gives the connection to the EL equations.

Lemma 5.4. *The contributions to the EL equations $\sim |\vec{\xi}|^{-3} \log |\vec{\xi}|$ vanish if and only if the condition* (5.19) *holds.*

Proof. A direct computation shows that the terms of the form (5.18) contribute to the EL equations of the order $|\vec{\xi}|^{-3} \log |\vec{\xi}|$ unless (5.19) holds. Therefore, our task is to show that it is impossible to compensate a term of the form (5.18) by a generalized microlocal chiral transformation. It clearly suffices to consider the homogeneous setting in the high-frequency limit as introduced in [7, §7.9]. Transforming to momentum space, the contribution (5.18) corresponds to the distribution

$$\gamma^i\, h_{ij}\, k^j\, \delta''(k^2)\, \Theta(-k^0) \,. \tag{5.23}$$

Having only three generations to our disposal, such a contribution would necessarily give rise to error terms of the form

$$\frac{1}{m^2} \gamma^i\, h_{ij}\, k^j\, \delta'(k^2)\, \Theta(-k^0) \qquad \text{or} \qquad \frac{1}{m^4} \gamma^i\, h_{ij}\, k^j\, \delta(k^2)\, \Theta(-k^0) \,.$$

These error terms as large as the shear contributions by local axial transformation as analyzed in [7, §7.8], causing problems in the EL equations (for details see [7, §7.8 and Appendix C]). Instead of going through these arguments again, we here rule out (5.23) with the following alternative consideration: In order to generate the contribution (5.23), at least one of the Dirac seas would have to be perturbed by a contribution with the scaling

$$\frac{1}{m^4}\, \gamma^i\, h_{ij}\, k^j\, \delta(k^2 - m_\alpha^2)\, \Theta(-k^0) \,.$$

Due to the factor k^j, this perturbation is by a scaling factor Ω larger than the perturbations considered in [7, §7.9]. Thus one would have to consider a transformation of the form (cf. [7, eq. (7.58)])

$$U = \exp{(iZ)} \qquad \text{with} \qquad Z = \mathcal{O}(\Omega^0) \,.$$

This transformation does not decay in Ω and thus cannot be treated perturbatively. Treating it non-perturbatively, the resulting shear contributions violate the EL equations. $\square$

Combining Lemmas 5.3 and 5.4 gives Proposition 5.1.

5.2. **The Field Tensor Terms.** We now come to the analysis of the contributions to the fermionic projector

$$\begin{aligned}\chi_L\, P(x,y) &\asymp \frac{1}{4}\,\chi_L\, \not{\xi} \int_x^y F_L^{ij}\,\gamma_i\gamma_j\, T^{(0)} - \chi_L\, \xi_i \int_x^y [0,1\,|\,0]\, F_L^{ij}\,\gamma_j\, T^{(0)} \\ &= \frac{1}{2}\,\chi_L \int_x^y (2\alpha-1)\,\xi_i\, F_L^{ij}\,\gamma_j\, T^{(0)} + \frac{i}{4}\,\chi_L \int_x^y \epsilon_{ijkl}\, F_L^{ij}\,\xi^k\,\gamma^5\gamma^l\, T^{(0)}\;, \end{aligned} \tag{5.24}$$

which we refer to as the *field tensor terms* (see [2, Appendix A], [3, Appendix A] and [7, Appendix B]; note that here we only consider the *phase-free* contributions, to which gauge phases can be inserted according to the rules in [3]). In [7], the field tensor terms were disregarded because they vanish when the Dirac matrices are contracted with outer factors ξ. Now we will analyze the field tensor terms in the ι-formalism introduced in §2.7. This will give additional constraints for the form of the admissible gauge fields (see relation (5.37) below).

In this section, the corrections in τ_{reg} are essential. It is most convenient to keep the terms involving τ_{reg} in all computations. We assume that we evaluate weakly for such a small vector ξ that we are in case **(ii)** in (3.35) (this will be discussed in Section 7). It then suffices to consider the sector-diagonal elements of the closed chain. Moreover, by restricting attention to the first or second isospin component, we can compute the spectral decomposition of the closed chain in the neutrino sector ($n=1$) and the chiral sector ($n=2$). separately. For a uniform notation, we introduce the notation

$$M_n^{(l)} = \begin{cases} L^{(l)}_{[0]} & \text{if } n=1 \\ T^{(l)}_{[0]} & \text{if } n=2 \end{cases}$$

with $L^{(l)}_\circ$ as given by (3.9). Then the unperturbed eigenvalues are given by

$$\lambda_{nL-} = 9\, T^{(-1)}_{[0]} \overline{M_n^{(0)}}\;, \qquad \lambda_{nR-} = 9\, M_n^{(-1)} \overline{T^{(0)}_{[0]}}\;.$$

Moreover, using the calculations

$$\frac{\overline{\lambda_{nL-}}}{|\lambda_{nL-}|}\,\chi_L P(x,y) = 3i\,\chi_L\,\not{\xi}\; \frac{M_n^{(0)}\overline{T^{(-1)}_{[0]}}}{\big|T^{(-1)}_{[0]} M_n^{(0)}\big|}\; T^{(-1)}_{[0]} = 3i\,\chi_L\,\not{\xi}\; \frac{\big|T^{(-1)}_{[0]}\big|}{\big|M_n^{(0)}\big|}\; M_n^{(0)}$$

$$\frac{\overline{\lambda_{nR-}}}{|\lambda_{nR-}|}\,\chi_R P(x,y) = 3i\,\chi_R\,\not{\xi}\; \frac{\big|M_n^{(-1)}\big|}{\big|T^{(0)}_{[0]}\big|}\; T^{(0)}_{[0]}$$

in (4.3), we can write the EL equations as

$$\Big(\Delta|\lambda_{nL-}| - \frac{1}{4} \sum_{n',c'} \Delta|\lambda_{n'c'-}| \Big) \frac{\big|T^{(-1)}_{[0]}\big|}{\big|M^{(0)}_n\big|}\, M^{(0)}_n = 0 \tag{5.25}$$

$$\Big(\Delta|\lambda_{nR-}| - \frac{1}{4} \sum_{n',c'} \Delta|\lambda_{n'c'-}| \Big) \frac{\big|M^{(-1)}_n\big|}{\big|T^{(0)}_{[0]}\big|}\, T^{(0)}_{[0]} = 0\,. \tag{5.26}$$

Note that in the case $\tau_{\text{reg}} = 0$, these equations reduce to our earlier conditions (4.5) and (4.6).

Our task is to analyze how (5.24) influences the eigenvalues λ_{nc-} of the closed chain.

Lemma 5.5. *The field tensor terms* (5.24) *contribute to the eigenvalues* λ_{nc-} *by*

$$\lambda_{nL-} \asymp \frac{3i}{2} \int_x^y (2\alpha-1)\; \text{Tr}_{\mathbb{C}^2} \Big(I_n\, \hat{F}^{ij}_L\, \check{\xi}_i \,\big(\iota^{(-1)}_{[0]}\big)_j \Big)\, T^{(0)}_{[0]} \overline{M^{(0)}_n} \tag{5.27}$$

$$\qquad + \frac{3}{4} \int_x^y \text{Tr}_{\mathbb{C}^2} \Big(I_n\, \epsilon_{ijkl}\, \hat{F}^{ij}_L\, \check{\xi}^k \,\big(\iota^{(-1)}_{[0]}\big)^l \Big)\, T^{(0)}_{[0]} \overline{M^{(0)}_n} + (\deg < 2) \tag{5.28}$$

$$\lambda_{nR-} \asymp \frac{3i}{2} \int_x^y (2\alpha-1)\; \text{Tr}_{\mathbb{C}^2} \Big(I_n\, \hat{F}^{ij}_R\, \check{\xi}_i \,\big(\iota^{(-1)}_{[0]}\big)_j \Big)\, M^{(0)}_n \overline{T^{(0)}_{[0]}} \tag{5.29}$$

$$\qquad - \frac{3}{4} \int_x^y \text{Tr}_{\mathbb{C}^2} \Big(I_n\, \epsilon_{ijkl}\, \hat{F}^{ij}_R\, \check{\xi}^k \,\big(\iota^{(-1)}_{[0]}\big)^l \Big)\, M^{(0)}_n \overline{T^{(0)}_{[0]}} + (\deg < 2)\,. \tag{5.30}$$

Proof. We first consider the effect of a left-handed field on the left-handed eigenvalues. Every summand in (5.24) involves a factor $\xi T^{(0)}$. As the factor $\iota^{(0)}$ gives no contribution (see (2.66)), we regularize (5.24) in the ι-formalism by

$$\chi_L\, P(x,y) \asymp \frac{1}{2}\, \chi_L \int_x^y (2\alpha-1)\, \check{\xi}_i\, \hat{F}^{ij}_L\, \gamma_j\, T^{(0)}_{[0]} + \frac{i}{4}\, \chi_L \int_x^y \epsilon_{ijkl}\, \hat{F}^{ij}_L\, \check{\xi}^k\, \gamma^5 \gamma^l\, T^{(0)}_{[0]} \tag{5.31}$$

(where the hat again denotes the partial trace). For computing the effect on the eigenvalues, we first multiply by the vacuum fermionic projector $P_0(y,x)$ to form the closed chain. Then we multiply by powers of the vacuum chain (2.69) and take the trace. Since the number of factors ι in (2.69) always equals the number of factors $\hat{\xi}$, and taking into account that (5.31) vanishes when contracted with a factor $\hat{\xi}$, we conclude that the factor $P_0(y,x)$ must contain a factor ι. In view of (2.66), this means that we only need to take into account the contribution $P_0(y,x) \asymp -3i\, \overline{\not{\iota}^{(-1)}_{[0]}\; L^{(0)}_{[0]}}$. We thus obtain

$$\begin{aligned} \chi_L\, A_{xy} \asymp\; & \frac{3i}{2}\, \chi_L \int_x^y (2\alpha-1)\, \check{\xi}_i\, \hat{F}^{ij}_L\, \gamma_j\, T^{(0)}_{[0]}\; \overline{\not{\iota}^{(-1)}_{[0]}\; M^{(0)}_n} \\ & - \frac{3}{4}\, \chi_L \int_x^y \epsilon_{ijkl}\, \hat{F}^{ij}_L\, \check{\xi}^k\, \gamma^5 \gamma^l\, T^{(0)}_{[0]}\; \overline{\not{\iota}^{(-1)}_{[0]}\; M^{(0)}_n}\,. \end{aligned}$$

Since the last Dirac factor involves ι, this contribution vanishes when multiplied by the first summand in (2.69). Thus our field tensor term only influences the eigenvalue λ_{nL-}. A short calculation gives (5.27) and (5.28). Similarly, a right-handed field only influences the corresponding right-handed eigenvalues by (5.29) and (5.30). The result follows by linearity. $\square$

Before going on, we remark that the above contributions do not appear in the standard formalism of the continuum limit, where all factors ξ which are contracted to

macroscopic functions are treated as outer factors. In order to get back to the standard formalism, one can simply impose that $F_{ij}\check{\xi}^i\iota^j = 0$. However, this procedure, which was implicitly used in [7], is not quite convincing because it only works if the regularization is adapted locally to the field tensor. If we want to construct a regularization which is admissible for any field tensor (which should of course satisfy the field equations), then the contributions by Lemma 5.5 must be taken into account.

Corollary 5.6. *Introducing the macroscopic functions*

$$a_{n\,L/R} = \frac{3i}{4} \int_x^y (2\alpha - 1)\; \mathrm{Tr}_{\mathbb{C}^2} \Big(I_n\, \hat{F}^{ij}_{L/R}\, \check{\xi}_i \,\big(\iota^{(-1)}_{[0]}\big)_j \Big) \tag{5.32}$$

$$\pm \frac{3}{8} \int_x^y \mathrm{Tr}_{\mathbb{C}^2} \Big(I_n\, \epsilon_{ijkl}\, \hat{F}^{ij}_{L/R}\, \check{\xi}^k \,\big(\iota^{(-1)}_{[0]}\big)^l \Big) \,, \tag{5.33}$$

the absolute values of the eigenvalues are perturbed by the field tensor terms (5.24) *according to*

$$\Delta|\lambda_{nL-}| = \frac{\big|M^{(0)}_n\big|}{\big|T^{(-1)}_{[0]}\big|} \Big(a_{nL}\, T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} + \overline{a_{nL}}\, T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} \Big)$$

$$\Delta|\lambda_{nR-}| = \frac{\big|T^{(0)}_{[0]}\big|}{\big|M^{(-1)}_n\big|} \Big(a_{nR}\, M^{(0)}_n \overline{M^{(-1)}_n} + \overline{a_{nR}}\, M^{(-1)}_n \overline{M^{(0)}_n} \Big) \,.$$

Proof. Writing the result of Lemma 5.5 as

$$\Delta\lambda_{nL-} = 2a_{nL}\, T^{(0)}_{[0]} \overline{M^{(0)}_n} \,, \qquad \Delta\lambda_{nR-} = 2a_{nR}\, M^{(0)}_n \overline{T^{(0)}_{[0]}} \,,$$

we obtain

$$\Delta|\lambda_{nL-}| = \frac{1}{\big|T^{(-1)}_{[0]} \overline{M^{(0)}_n}\big|}\, \mathrm{Re} \Big(a_{nL}\, T^{(0)}_{[0]} \overline{M^{(0)}_n} M^{(0)}_n \overline{T^{(-1)}_{[0]}} \Big) = \frac{\big|M^{(0)}_n\big|}{\big|T^{(-1)}_{[0]}\big|}\, \mathrm{Re} \Big(a_{nL}\, T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} \Big) \,.$$

The calculation for $\Delta|\lambda_{nR-}|$ is analogous. ☐

After these preparations, we are ready to analyze the EL equations (5.25) and (5.26). We begin with the case $\tau_{\mathrm{reg}} = 0$. Then we can set $M^{(l)}_n = T^{(l)}_{[0]}$, giving the conditions (4.5), where now

$$\mathcal{K}_{nc} = \Delta|\lambda_{nL-}|\, \frac{\big|T^{(-1)}_{[0]}\big|}{\big|T^{(0)}_n\big|}\, M^{(0)}_n = a_{nc}\, T^{(0)}_{[0]} T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} + \overline{a_{nc}}\, T^{(0)}_{[0]} T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} \,.$$

This formula can be simplified further with the integration-by-parts rules. Namely, applying (2.40), we obtain

$$0 = \nabla \Big(T^{(0)}_{[0]} T^{(0)}_{[0]} \overline{T^{(0)}_{[0]}} \Big) = 2\, T^{(0)}_{[0]} T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} + T^{(0)}_{[0]} T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} \,,$$

Using this relation, we conclude that

$$\mathcal{K}_{nc} = -\Big(2a_{nc} - \overline{a_{nc}} \Big)\, T^{(0)}_{[0]} T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} = -\Big(\mathrm{Re}(a_{nc}) + 3\, \mathrm{Im}(a_{nc}) \Big)\, T^{(0)}_{[0]} T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} \,.$$

If any non-trivial gauge field is present, the four macroscopic functions a_{nc} will not all be the same (note that even for a vectorial field which is acts trivially on the isospin

index, the contribution term (5.33) has opposite sign for a_{nL} and a_{nR}). This implies that (4.5) can be satisfied only if we impose the regularization condition

$$T^{(0)}_{[0]} T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} = 0 \,, \tag{5.34}$$

which must hold in a weak evaluation on the light cone.

In order to compute the effect of τ_{reg}, we first note that the perturbations $\Delta|\lambda_{2c-}|$ do not involve τ_{reg} (as is obvious from Corollary 5.6). Moreover, the contribution of these eigenvalues to (5.25) and (5.26) for $n = 2$ is independent of τ_{reg}. In view of (5.34), these contributions drop out of the EL equations. Next, the eigenvalue λ_{1L-} contributes to (5.25) and (5.26) for $n = 2$ by

$$-\frac{1}{4}\, \Delta|\lambda_{1L-}| \frac{|T^{(-1)}_{[0]}|}{|T^{(0)}_{[0]}|}\, T^{(0)}_{[0]} = \frac{|L^{(0)}_{[0]}|}{|T^{(0)}_{[0]}|}\, T^{(0)}_{[0]} \Big(a_{nL}\, T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} + \overline{a_{nL}}\, T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} \Big). \tag{5.35}$$

This is in general non-zero. Thus in order to allow for left-handed gauge fields in the neutrino sector, we need to impose additional conditions on the regularization functions. The simplest method is to impose that

$$T^{(0)}_{[R,0]} = (\deg \leq 1) \cdot \mathcal{O}\big((m\varepsilon)^{p_{\text{reg}}}\big) \qquad \text{pointwise}\,. \tag{5.36}$$

Then $L^{(0)}_{[0]} = T^{(0)}_{[0]}$, and (5.35) again vanishes as a consequence of (5.35). We remark that (5.36) could be replaced by a finite number of equations to be satisfied in a weak evaluation on the light cone. But as these equations are rather involved, we here prefer the stronger pointwise condition (5.36). We also note that, in contrast to the condition (5.34) for the regularization of ordinary Dirac seas, the relation (5.36) imposes a constraint only on the right-handed high-energy states.

It remains to consider the terms involving $T^{(-1)}_{[R,0]}$. These are $\Delta|\lambda_{1R-}|$ as well as the factor $|M^{(-1)}_n|$ in (5.26) in case $n = 1$. Collecting all the corresponding contributions to the EL equations, we get a finite number of equations to be satisfied in a weak evaluation on the light cone. Again, we could satisfy all these equations by imposing suitable conditions on the regularization. However, these additional conditions would basically imply that $T^{(-1)}_{[R,0]} = 0$ vanishes, meaning that there are no non-trivial regularization effects. For this reason, our strategy is not to impose any more regularization conditions. Then the EL equations (5.25) and (5.26) are satisfied if and only if there is no right-handed gauge field in the neutrino sector and if the vectorial component is trace-free,

$$\text{Tr}(I_1 F_R) = 0 \qquad \text{and} \qquad \text{Tr}(F_L + F_R) = 0 \,, \tag{5.37}$$

because only under these conditions all the equations in involving $T^{(-1)}_{[R,0]}$ or $\overline{T^{(-1)}_{[R,0]}}$ vanish.

6. Projection on the Dynamical Degrees of Freedom

6.1. **Preparatory Considerations.** Let us briefly review our general strategy for deriving the effective field equations. Our starting point is the fermionic projector of the vacuum, being composed of solutions of the free Dirac equation (cf. (2.46))

$$(i\partial\!\!\!/ - mY)\Psi = 0 \,.$$

As explained in §4.3, a wave function Ψ gives rise to a contribution to the fermionic projector of the form (cf. (2.52))

$$-\frac{1}{2\pi}\overline{\Psi}(x)\Psi(x)\,, \tag{6.1}$$

which enters the EL equations. Our method for satisfying the EL equations is to introduce a suitable potential into the Dirac equation (see (2.47))

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\Psi = 0\,. \tag{6.2}$$

After compensating the resulting logarithmic singularities by a microlocal chiral transformation (see §4.4), we can hope that the remaining smooth contributions to the fermionic projector by the Yang-Mills current can compensate the contribution by the Dirac current (6.1). This general procedure is worked out for an axial potential in [7].

In the present setting of two sectors, the situation is more involved because the potential $\mathcal{B}$ in (6.2) must satisfy the additional constraints (see (3.29) and the conditions arising from the structural contributions in Section 5). In order to explain the basic difficulty, let us consider the simplified situation where the potential must be vectorial and isospin diagonal; more precisely,

$$\mathcal{B} = A\!\!\!/\,\sigma^3 \tag{6.3}$$

with a vector field A. This constraint for the form of the potential implies that the YM current must also be of a special form. As a consequence, we cannot expect to compensate an arbitrary contribution of the form (6.2). The best we can hope for is to compensate the vectorial and isospin diagonal component of (6.2), suggesting the resulting field equation

$$\partial_{jk}A^k - \Box A_j = c\,\overline{\Psi}\gamma_j\Psi \tag{6.4}$$

with a coupling constant c. However, this choice of field equation is not unique. More generally, one could try to compensate another component of (6.2), leading to the field equation

$$\partial_{jk}A^k - \Box A_j = \overline{\Psi}(\vec{c}\,\vec{\sigma})\,\gamma_j\Psi \tag{6.5}$$

with a fixed vector $\vec{c} \in \mathbb{R}^3$. In general terms, one can say that, due to the constraints, we can satisfy the EL equations only "in the direction of" the admissible variations of the Dirac operator. This situation bears some some similarity with the Lagrange multiplier method as used in minimization problems with constraints. But for the Lagrange multiplier method, one needs to know what the "orthogonal complement" of the admissible variations are. For our variational principle, however, there is no notion of an "orthogonal complement". In order to give this notion a meaning, one would have to compute how the action changes in the direction of non-admissible variations. It is not clear how this computation could be carried out, because it seems to involve unknown regularization details. Even if such computations could be carried out, they would certainly be long and tedious and seem impractical.

As a way out, additional input is obtained by demanding that the resulting system of equations should be consistent in the sense that it admits non-trivial solutions. In order to explain the method in the above simplified example (6.3), we take the

divergence of (6.5) and use (6.2) together with (6.3) to obtain

$$\begin{aligned} 0 &= \overline{\partial_j \Psi}(\vec{c}\,\vec{\sigma})\gamma_j \Psi + \overline{\Psi}(\vec{c}\,\vec{\sigma})\gamma_j \partial_j \Psi \\ &= i\left(\overline{(i\partial\!\!\!/ - m)\Psi}(\vec{c}\,\vec{\sigma})\gamma_j \Psi - \overline{\Psi}(\vec{c}\,\vec{\sigma})(i\partial\!\!\!/ - m)\Psi\right) \\ &= -i\left(\overline{\Psi} A\!\!\!/ \sigma^3 (\vec{c}\,\vec{\sigma})\gamma_j \Psi - \overline{\Psi}(\vec{c}\,\vec{\sigma}) A\!\!\!/ \sigma^3 \Psi\right). \end{aligned} \tag{6.6}$$

The resulting equation admits non-trivial solutions only if

$$\sigma^3\,(\vec{c}\,\vec{\sigma}) = (\vec{c}\,\vec{\sigma})\,\sigma^3\,.$$

This is our consistency condition. It reduces the freedom to choosing the field equations within the two-parameter family $\vec{c} = c\mathbb{1} + d\sigma^3$. In order to understand the significance of such consistency conditions, one should keep in mind that more complicated systems of equations give rise to many more consistency conditions. For instance, additional consistency conditions are obtained by demanding that the total energy-momentum tensor be divergence free. More complicated consistency conditions arise if off-diagonal potentials and the MNS mixing are taken into account. Therefore, it is to be expected that consistency conditions drastically reduce the freedom in formulating the field equations.

Since it is not know how to systematically derive all consistency conditions, our method is to try to recover the field equations as the Euler-Lagrange equations corresponding to an underlying "effective Lagrangian." We now explain this method in the above simplified example, whereas the general construction will be given in the next section. We first note that the Dirac equation (6.2) can be obtained from the Dirac action action

$$\int \overline{\Psi}(i\partial\!\!\!/ + \mathcal{B} - mY)\Psi\, d^4x \tag{6.7}$$

by varying Ψ. In order to derive the field equations, we choose a kinetic term for the bosonic potential,

$$\int (\partial_j A_k)\,(\partial^j A^k)\, d^4x\,, \tag{6.8}$$

which is chosen such that varying A gives the left side of (6.5). The "effective action" is now chosen as a linear combination of (6.7) and (6.8). As an overall constant is irrelevant, this method only involves one free constant. The resulting field equation is necessarily of the form (6.4). Working with the effective action has the advantage that the conserved quantities obtained via Noether's theorem immediately guarantee consistency of the equations (for example, (6.6) is a consequence of current conservation).

6.2. **Projection on the Dynamical Degrees of Freedom.** Quantities like currents and fields take values in the Hermitian 6×6-matrices and have a left- and right-handed component. Thus, taking the direct sum of the two chiral components, it is useful to introduce the real vector space

$$\mathfrak{S}_6 := \text{Symm}(\mathbb{C}^6) \oplus \text{Symm}(\mathbb{C}^6)\,, \tag{6.9}$$

where $\text{Symm}(\mathbb{C}^6)$ denotes the Hermitian 6×6-matrices. For example, the Dirac current (4.23) can be regarded as an element of $\mathfrak{S}_6$,

$$\mathcal{J} := (J_L, J_R) \in \mathfrak{S}_6$$

(here we disregard the tensor indices, which will be included later in a straightforward way). Similarly, the Yang-Mills current (4.22) can be regarded as an element

of $\mathfrak{S}_6$. However, it can take values only in a subspace of $\mathfrak{S}_6$, as we now make precise. We denote the local gauge group corresponding to the admissible gauge potentials by $\mathcal{G} \subset \mathrm{U}(6)_L \times \mathrm{U}(6)_R$ and refer to it as the *dynamical gauge group* (recall that in §3.2 we found the group (3.28), and that taking into account the additional constraints encountered in Section 5, the dynamical gauge group is a proper subgroup of (3.28)). The *dynamical gauge potentials* are elements of the corresponding Lie algebra $\mathfrak{g} = T_e\mathcal{G}$, which can be identified with a subspace of $\mathfrak{S}_6$, the so-called *dynamical subspace*. The dynamical potentials and corresponding Yang-Mills currents take values in the dynamical subspace,

$$\mathcal{A} := (A_L, A_R) \in \mathfrak{g} \qquad \text{and} \qquad (j_L, j_R) \in \mathfrak{g}\,. \tag{6.10}$$

In our context, the usual *Dirac Lagrangian* takes the form

$$\mathcal{L}_{\mathrm{Dirac}} = \overline{\Psi}(i\partial\!\!\!/ + \mathcal{B} - mY)\Psi\,. \tag{6.11}$$

By varying the Dirac wave functions, one gets the Dirac equation (2.47). Conversely, the Dirac equation (2.47) forces us to choose the Dirac Lagrangian according to (6.11). The point is that by varying the Yang-Mills potential and the gravitational field, the Dirac Lagrangian also determines how the Dirac spinors couple to the bosonic fields. Our goal is to rewrite certain components of (3.3) in a way which is compatible with the coupling of the Dirac spinors as determined by the structure of the Dirac action. First, we need to modify (6.11) in order to build in that the fermionic projector involves the partial trace (2.53), implying that the coupling of the fermions to the bosonic fields is necessarily described after taking the partial trace. To this end, we replace (6.11) by

$$\mathcal{L}_{\mathrm{Dirac}} = \mathrm{Tr}_{\mathbb{C}^6}\left(\uparrow \overline{\Psi}(i\partial\!\!\!/ + \mathcal{B} - mY)\Psi\right), \tag{6.12}$$

where $\uparrow$ denotes the matrix which takes the partial trace over the generations and acts trivially on the isospin index,

$$\uparrow = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \otimes \mathbb{1}_{\mathbb{C}^2}\,.$$

Varying the wave function in (6.12) yields the equation $(i\partial\!\!\!/ + \mathcal{B} - mY)\Psi\uparrow$, which is weaker than the original Dirac equation because the matrix $\uparrow$ is singular (more precisely, we only obtain that the sum of the Dirac equations for the three generations must be satisfied). Thus it is not possible to derive the Dirac equations from the effective action. But the EL equations of the effective action are compatible with the Dirac equation. This ensures that our system of field equations and Dirac equations is consistent, and that all the conservation laws obtained from Noether's theorem are respected.

In order to restrict attention to the essence of (6.12), we first note that taking the partial trace gives rise to an operator

$$\hat{}\,:\; \mathfrak{S}_6 \to \mathfrak{S}_2 \subset \mathfrak{S}_6\,,$$

where in the last inclusion we regard a symmetric 2×2-matrix as a 6×6-matrix which acts trivially on the generation index (and $\mathfrak{S}_2$ denotes similar to (6.9) the chiral Hermitian 2×2-matrices). The coupling of the fermionic wave functions to the chiral potentials in (6.12) is described by the bilinear form

$$\langle .|.\rangle \;:\; \mathfrak{S}_2 \times \mathfrak{g} \to \mathbb{R} \;:\; \big(\hat{\jmath}, \mathcal{A}\big) \mapsto \mathrm{Tr}_{\mathbb{C}^6}(\hat{J}_L A_L + \hat{J}_R A_R)\,, \tag{6.13}$$

where $J_{L/R}$ is the Dirac current (4.23). We refer to $\langle .|. \rangle$ as the *fermion-boson pairing*. Next, we introduce the following subspaces of $\mathfrak{S}_2$,

$$\mathfrak{I} = \hat{\mathfrak{g}} \qquad \text{and} \qquad \mathfrak{N} = \{ \hat{\mathfrak{J}} \mid \langle \hat{\mathfrak{J}} | \mathcal{A} \rangle = 0 \quad \forall \mathcal{A} \in \mathfrak{g} \} ,$$

giving rise to the direct sum decomposition

$$\mathfrak{S}_2 = \mathfrak{I} \oplus \mathfrak{N} .$$

The space $\mathfrak{I}$, referred to as the *interacting subspace*, has the interpretation as the components of the Dirac current (always after taking the partial trace) which couple to the bosonic field. The *non-interacting subspace* $\mathfrak{N}$, on the other hand, does not take part in the interaction.

We next analyze how the potentials and currents enter the EL equations (6.11). As explained in §4.1, we work with the stronger conditions (4.13). We consider the contributions to degree four on the light cone after compensating the logarithmic poles and evaluating weakly on the light cone. Writing the left- and right-handed components again as a direct sum, we have symbolically

$$\big(\mathcal{R}_L(x,y), \mathcal{R}_R(x,y) \big) \asymp \Big(\Phi_J(\hat{\mathfrak{J}}_j) + \Phi_B\big(\partial_{jk} \mathcal{A}^k - \Box \mathcal{A}_j \big) + \Phi_M(\mathcal{A}_j) \Big) \, i \xi^j , \tag{6.14}$$

where the linear operators Φ_J, Φ_B and Φ_M describe how the fermionic and bosonic currents as well as the mass terms enter the matrices $\mathcal{R}_L$ and $\mathcal{R}_R$,

$$\Phi_J \,:\, \mathfrak{S}_2 \rightarrow \mathfrak{S}_2 \,, \qquad \Phi_B, \Phi_M \,:\, \mathfrak{g} \rightarrow \mathfrak{S}_2 \,.$$

We want to deduce from (6.14) a set of equations in which the current $\mathfrak{J}$ enters only in terms of the fermion-boson pairing (6.13). The obvious idea is to multiply (6.14) by the inverse of Φ_J and then take the fermion-boson pairing with $\mathcal{A}^j$. Unfortunately, this method does not work because Φ_J is in general not invertible. But the method can be realized if we multiply instead by a matrix Ξ_J, which is uniquely determined by imposing that the non-interacting subspace should not enter the effective Lagrangian. More precisely, we make the following assumptions:

(a) The mapping $\Phi_J|_{\mathfrak{I}} \,:\, \mathfrak{I} \rightarrow \mathfrak{S}_2$ is injective.
(b) $\Phi_J(\mathfrak{I}) \cap \Phi_J(\mathfrak{N}) = \{0\}$.
(c) The sets $\Phi_B(\mathfrak{g})$ and $\Phi_M(\mathfrak{g})$ are contained in $\Phi_J(\mathfrak{S}_2)$.

Under these assumptions, we can introduce a mapping

$$\Xi_J : \Phi_J(\mathfrak{S}_2) \rightarrow \mathfrak{I} \subset \mathfrak{S}_2$$

as follows. Suppose that $v \in \Phi_J(\mathfrak{S}_2)$. According to (b), we can uniquely decompose v as

$$v = v_{\mathfrak{I}} + v_{\mathfrak{N}} \qquad \text{with} \qquad v_{\mathfrak{I}} \in \Phi_J(\mathfrak{I}) \text{ and } v_{\mathfrak{N}} \in \Phi_J(\mathfrak{N}) \,.$$

Using (a), there is a unique vector $u \in \mathfrak{I}$ with $\Phi_J(u) = v_{\mathfrak{I}}$. Setting $\Xi_J(v) = u$ defines the desired linear mapping. It has the properties

$$\Xi_J|_{\Phi_J(\mathfrak{I})} = (\Phi_J|_{\mathfrak{I}})^{-1} \qquad \text{and} \qquad \Xi_J|_{\Phi_J(\mathfrak{N})} = 0 \,.$$

Next, according to (c), the vector (6.14) lies in the image of Φ_J. Hence we may multiply (6.14) by Ξ_J. Taking the fermion-boson pairing with a potential $\mathcal{A}' \in \mathfrak{g}$ gives the necessary conditions

$$\Big\langle \big(\Xi_J \circ \Phi_J\big)(\hat{\mathfrak{J}}_j) + \big(\Xi_J \circ \Phi_B\big)\big(\partial_{jk}\mathcal{A}^k - \Box \mathcal{A}_j\big) + \big(\Xi_J \circ \Phi_M\big)(\mathcal{A}_j) \Big| \mathcal{A}' \Big\rangle = 0 \quad \forall \mathcal{A}' \in \mathfrak{g}. \tag{6.15}$$

These conditions can be further simplified: If $\hat{\mathfrak{J}}$ lies in the interacting subspace, then $(\Xi_J \circ \Phi_J)(\hat{\mathfrak{J}}) = \hat{\mathfrak{J}}$. On the other hand, if $\hat{\mathfrak{J}}$ lies in the non-interacting subspace, then $(\Xi_J \circ \Phi_J)(\hat{\mathfrak{J}}$ vanishes. In the latter case, the fermion-boson pairing $(\hat{\mathfrak{J}}|\mathcal{A}') = 0$ also vanishes. Hence we may replace (6.15) by

$$\Big\langle \hat{\mathfrak{J}}_j + \big(\Xi_J \circ \Phi_B\big)\big(\partial_{jk}\mathcal{A}^k - \Box \mathcal{A}_j\big) + \big(\Xi_J \circ \Phi_M\big)(\mathcal{A}_j) \,\Big|\, \mathcal{A}' \Big\rangle = 0 \qquad \forall \mathcal{A}' \in \mathfrak{g} \,. \tag{6.16}$$

With the above construction, we have reduced the EL equations (6.11) to equations corresponding to the dynamical degrees of freedom as described by the subspace $\mathfrak{g} \subset \mathfrak{S}_6$. We refer to this procedure as the *projection on the dynamical degrees of freedom*.

The above construction can be adapted in a straightforward way to the gravitational field: We first write the contribution of the energy-momentum and the Ricci tensor to the EL equations in analogy to (6.14) symbolically as

$$\big(\mathcal{R}_L(x,y), \mathcal{R}_R(x,y)\big) \asymp \Big(\Phi_T(\hat{T}_{jk}) + \Phi_{\text{curv}}(R_{jk})\Big)\, \xi^j \xi^k \,, \tag{6.17}$$

where Φ_T and Φ_{curv} are linear mappings

$$\Phi_T \,:\, \mathfrak{S}_2 \to \mathfrak{S}_2 \,, \qquad \Phi_{\text{curv}} \,:\, \mathbb{R} \to \mathfrak{S}_2 \,.$$

We introduce $\mathfrak{I}_{\text{curv}} = <(\mathbb{1}, \mathbb{1})> \subset \mathfrak{S}_2$ and set $\mathfrak{N}_{\text{curv}} = \{\hat{\mathfrak{J}} \mid \langle \hat{\mathfrak{J}} | (\mathbb{1}, \mathbb{1}) \rangle = 0\}$, giving rise to the direct sum decomposition

$$\mathfrak{S}_2 = \mathfrak{I}_{\text{curv}} \oplus \mathfrak{N}_{\text{curv}} \,.$$

Similar to the above conditions (a)–(c), we make the following assumptions.

(d) The mapping $\Phi_T|_{\mathfrak{I}_{\text{curv}}} \,:\, \mathfrak{I}_{\text{curv}} \to \mathfrak{S}_2$ is injective.
(e) $\Phi_T(\mathfrak{I}_{\text{curv}}) \cap \Phi_T(\mathfrak{N}_{\text{curv}}) = \{0\}$.
(f) $\Phi_{\text{curv}}(\mathbb{R}) \subset \Phi_T(\mathfrak{S}_2)$.

Under these assumption, there is a unique linear operator $\Xi_T : \Phi_T(\mathfrak{S}_2) \to \mathfrak{I}_{\text{curv}}$ with the properties

$$\Xi_T|_{\Phi_T(\mathfrak{I}_{\text{curv}})} = (\Phi_T|_{\mathfrak{I}_{\text{curv}}})^{-1} \qquad \text{and} \qquad \Xi_T|_{\Phi_T(\mathfrak{N}_{\text{curv}})} = 0 \,.$$

Multiplying (6.17) by Ξ_T and taking the fermion-boson pairing with $(\mathbb{1}, \mathbb{1})$, we obtain the necessary condition

$$\big\langle \hat{T}_{jk} + \big(\Xi_T \circ \Phi_{\text{curv}}\big)(R_{jk}) \,\big|\, (\mathbb{1}, \mathbb{1}) \big\rangle = 0 \,. \tag{6.18}$$

6.3. **The Effective Lagrangian.** In the equation (6.16) we arranged that the Dirac current couples to the bosonic potentials just as required from the Dirac action (6.11). Thus in order for this equation to be of variational form, we merely need to assume that the terms involving the bosonic current and the mass term are variational derivatives of a corresponding Lagrangian. A simple sufficient condition, which will always be satisfied in our applications, is that the corresponding quadratic forms are symmetric. We thus make the following assumption:

(g) The bilinear forms

$$\big\langle (\Xi_J \circ \Phi_B)(.) \,|\, . \big\rangle \,:\, \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \qquad \text{and} \qquad \big\langle (\Xi_J \circ \Phi_M)(.) \,|\, . \big\rangle \,:\, \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \tag{6.19}$$

are symmetric.

Then we can introduce Yang-Mills Lagrangian by

$$\mathcal{L}_{\rm YM} = \frac{1}{2} \left\langle (\Xi_J \circ \Phi_B)\big(\partial_{jk}\mathcal{A}^k - \Box \mathcal{A}_j\big) \,\big|\, \mathcal{A}^j \right\rangle + \frac{1}{2} \left\langle (\Xi_J \circ \Phi_M)(\mathcal{A}) \,\big|\, \mathcal{A} \right\rangle$$

By adding irrelevant divergence terms, we can write this Lagrangian in the equivalent but more familiar form

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} \left\langle (\Xi_J \circ \Phi_B)\big(\partial_j \mathcal{A}_k - \partial_k \mathcal{A}_j\big) \,\big|\, (\partial^j \mathcal{A}^k - \partial^k \mathcal{A}^j)\right\rangle + \frac{1}{2} \left\langle (\Xi_J \circ \Phi_M)(\mathcal{A}) \,|\, \mathcal{A} \right\rangle .$$

Adding the Dirac Lagrangian (6.11) gives the effective action

$$\mathcal{S}_{\rm eff} = \int_M (\mathcal{L}_{\rm Dirac} + \mathcal{L}_{\rm YM})\, d^4x \,. \tag{6.20}$$

Varying Ψ gives the Dirac equation (2.47). Varying the Yang-Mills potential gives precisely the equations (6.16) with $\mathcal{A} \in \mathfrak{g}$.

In order to incorporate the field equation for the gravitational field into the effective Lagrangian, one replaces the integration measure in (6.20) by $\sqrt{-\det g}\, d^4x$ and the Dirac operator as usual by that in curved space-time. Moreover, one adds the Einstein-Hilbert action

$$\mathcal{S}_{\rm EH} = \int_M \left\langle (\Xi_T \circ \Phi_{\rm curv})(R) \,\big|\, (\mathbb{1}, \mathbb{1}) \right\rangle \sqrt{-\det g}\, d^4x \,, \tag{6.21}$$

where R denotes scalar curvature. One should keep in mind that this Lagrangian is determined only up to terms which contribute to the EL equations (3.2) to degree three or lower. In particular, if the Ricci tensor is a multiple of the metric, the term $R_{jk}\xi^j\xi^k$ in Lemma 4.12 is of degree one on the light cone, giving rise to a contribution which can be absorbed in the error term. In other words, to the considered degree four on the light cone, the Ricci tensor is determined only up to multiples of the metric. This gives precisely the freedom to add the cosmological Lagrangian

$$\int_M \frac{2\Lambda}{\kappa} \sqrt{-\det g}\, d^4x$$

for an arbitrary value of the cosmological constant Λ. In principle, the cosmological constant could be determined in our approach by evaluating the EL equations to degree three on the light cone. But this analysis goes beyond the scope of the present work.

We finally point out that the above method determines the effective Lagrangian and fixes the coupling constants (except for the cosmological constant). Thus the structure of the interaction is described completely by the underlying EL equations (3.3) corresponding to the causal action principle (1.3).

6.4. **General Remarks.** The projection on the dynamical degrees of freedom gives a method to extract from the EL equations (3.3) a system of field equations which are of variational form. The wish to rewrite the field equations as the critical points of an effective action is also motivated by the fact that we describe the interaction by the underlying causal action principle (1.3), and it seems natural to assume that the variational form of the equations should be preserved when taking the continuum limit. However, at present there is no mathematical method to get directly from the action (1.2) to the above effective action. Our methods make it necessary to first go to the EL equations (3.3) corresponding to (1.2). Analyzing these EL equations in the continuum limit gives relations in which the variational form is no longer apparent. By

demanding that the equations should be again of variational form, we found a unique way of rewriting the field equations as coming from an effective Lagrangian.

Although this procedure seems sufficient for all practical purposes, it is not quite satisfying from the theoretical point of view. There are two conceivable methods to improve the situation, which we mention as possible directions of future research:

(A) In order to prove that the EL equations (3.3) are again of variational form in the continuum limit, one could analyze systematically how the non-dynamical degrees of freedom contribute to the EL equations (examples for such non-dynamical degrees of freedom are the gauge potentials which were ruled out in §3.2 and §4, but there are many other non-dynamical variations which have not been considered so far). Similar as in the Lagrange multiplier method, it suffices to satisfy the EL equations in the directions transverse to the non-dynamical contributions. This analysis, which might make it necessary to impose certain conditions on the structure of the regularization, could give a detailed justification of the evaluations (6.16) and (6.18).

(B) It would be desirable to have a mathematical method which would allow to take some kind of "continuum limit" of the action (1.2) to obtain the effective action directly (i.e. without analyzing the EL equations). The general difficulty is that the causal action principle (1.3) involves constraints (in particular the condition that P should be a projector of fixed rank) which have no obvious correspondence in the continuum limit. This difficulty is also reflected in a quite different structure of the action principles: In (1.3) the action is minimized. For the effective action, a minimization makes no sense because the effective action is unbounded from below. Instead, one merely considers critical points of the effective action. This general difference must be taken into account in any approach towards relating (1.2) directly to an effective action.

7. The Field Equations for Chiral Gauge Fields

We now use the methods of Section 6 to compute the effective action for the coupling of the Dirac field to the gauge fields. In order to determine the dynamical subspace $\mathfrak{g} \subset \mathfrak{S}_6$ (defined before (6.10)), we first recall that in §3.2 we derived the admissible local gauge group (3.28) together with the representation of the gauge potentials (3.29). In Section 5, we obtained further restrictions for the gauge potentials. Namely, the analysis of the bilinear logarithmic terms in §5.1 revealed that the diagonal elements must satisfy the constraint (5.1). The field tensor terms in §5.37, on the other hand, gave us the two linear constraints (5.37) for the field tensor, which due to gauge symmetry we can also regard as constraints for the potentials. Putting these conditions together, we conclude that the dynamical gauge potentials must be of one of the two alternative forms

$$\mathcal{B} = \chi_R \begin{pmatrix} \not{A}_L^{11} & \not{A}_L^{12}\, U_{\text{MNS}}^* \\ \not{A}_L^{21}\, U_{\text{MNS}} & 0 \end{pmatrix} + \chi_L \begin{pmatrix} 0 & 0 \\ 0 & -\not{A}_L^{11} \end{pmatrix} \quad \text{or} \tag{7.1}$$

$$\mathcal{B} = \chi_R \begin{pmatrix} \not{A}_L^{11} & \not{A}_L^{12}\, U_{\text{MNS}}^* \\ \not{A}_L^{21}\, U_{\text{MNS}} & -\not{A}_L^{11} \end{pmatrix} + \chi_L \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.2}$$

The potentials of the form (7.1) do do not form a Lie algebra. This means that the structure of (7.1) is not preserved under local gauge transformations corresponding to the potentials of the form (7.1). As this seems to be inconsistent, we disregard

this case. We thus restrict attention to the remaining case (7.2), where $\mathfrak{g}$ is the Lie algebra su(2), which acts on the left-handed component of the spinors and involves a MNS mixing matrix. Hence

$$\mathfrak{I} = \{(\vec{\sigma}\vec{v}, 0) \mid \vec{v} \in \mathbb{R}^3\} \qquad \text{and} \qquad \mathfrak{N} = \{(a\,\mathbb{1}, B) \mid a \in \mathbb{R}, B \in \mathrm{Symm}(\mathbb{C}^2)\} .$$

Using the results of Section 4, it is straightforward to compute Φ_J to degree four on the light cone. Applying Lemma 4.4 and (4.24), we obtain

$$\mathcal{K}_L \asymp i\hat{J}^k_R\, \xi_k\, K_1 ,$$

and similarly for the right-handed component. Using this identity in (4.12) gives

$$\mathcal{R}_L \asymp i\xi_k \Big(\hat{J}^k_R - \frac{1}{4}\, \mathrm{Tr}_{\mathbb{C}^2}\big(\hat{J}^k_L + \hat{J}^k_R\big)\, \mathbb{1}_{\mathbb{C}^2} \Big)\, K_1 .$$

Using this formula in (6.14), we obtain

$$\Phi_J\big((\hat{J}_L, \hat{J}_R)\big) = \Big((\hat{J}_R, \hat{J}_L) - \frac{1}{4}\, \mathrm{Tr}_{\mathbb{C}^2}\big(\hat{J}_L + \hat{J}_R\big)\, (\mathbb{1}, \mathbb{1}) \Big)\, K_1 + (\deg < 4) . \tag{7.3}$$

A short calculation shows that the conditions (a) and (b) in §6.2 are satisfied. Furthermore, since the image of Φ_J consists of all trace-free matrices in $\mathfrak{S}_2$, it is obvious from (4.12) that the condition (c) is also satisfied. Moreover, in view of (7.3), the mapping Ξ_J is simply given by

$$\Xi_J((B_L, B_R)) = \frac{1}{2K_1} \sum_{\alpha=1}^{3} \mathrm{Tr}(\sigma_\alpha\, B_R)\; (\sigma_\alpha, 0) .$$

Next, the bilinear forms in (6.19) are symmetric, and thus the condition (g) in §6.3 also holds. We conclude that the field equations reduce to the three equations

$$0 = \mathrm{Tr}(\sigma_\alpha\, \mathcal{K}_R) , \qquad \alpha = 1, 2, 3 .$$

Here $\mathcal{K}_R$ contains the contributions by the left-handed Dirac and bosonic currents of Lemma 4.4, with the logarithmic poles compensated by the microlocal chiral transformation according to Proposition 4.7. We state the general structure of these field equations and discuss them afterwards.

Theorem 7.1. *Writing the* SU(2)*-gauge potentials in components*

$$A^\alpha_L = \frac{1}{2}\, \mathrm{Tr}(\sigma^\alpha A_L)$$

(and similarly for the currents), the field equations read

$$j^\alpha_L - M_\alpha^2\, A^\alpha_L = c_\alpha\, J^\alpha_L \;+\; (f_{[0]} * j_L)^\alpha + (f_{[2]} * A_L)^\alpha ,$$

where j_L and J_L are the currents (4.22) *and* (4.23)*, respectively. The mass parameters M_α and the coupling constants c_α satisfy the relations*

$$M_1 = M_2 \qquad \text{and} \qquad c_1 = c_2 . \tag{7.4}$$

Finally, the distributions $f_{[0]}$ and $f_{[2]}$ are convolution kernels.

Note that the relations (7.4) follow immediately from the symmetry under transformations of the two isospin components according to

$$\Psi \to \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \Psi .$$

Qualitatively speaking, this theorem can be understood similar to the results in [7]. Also, the calculations use exactly the same methods. In particular, the convolution kernels $f_{[0]}$ and $f_{[2]}$ are computed and interpreted just as in [7, §8.1 and §8.2]. In view of these similarities, we here omit the detailed computations and only point to two steps in the computations which are not quite straightforward. First, as mentioned after Proposition 4.6, the constants $\mathfrak{c}_0$ and $\mathfrak{c}_2$ are not determined by this proposition. Following the strategy used in [7, 7.9], we can fix these constants by minimizing $\mathfrak{c}_0$. Thus we choose the microlocal chiral transformation in such a way that the vectorial contribution (4.49) to the fermionic projector is as small as possible. Using this method, for a given regularization one can also compute the coupling constants and the masses similar as in [7, §8.6].

The second step which requires an explanation concerns the computation of the coupling constants and bosonic masses for a given regularization in the spirit of [7, §8.6]. Here one must distinguish the two cases **(i)** and **(ii)** in (3.35). Which of these cases applies depends crucially on the choice of the parameter p_{reg} in (3.33). In particular, by choosing p_{reg} sufficiently small (and thus the parameter τ_{reg} in (2.43) sufficiently large), we can arrange that we are in case **(ii)**. In order to keep the setting as general as possible, we deliberately left open which of the cases should be physically relevant. We found that all our computations up to and including Section 4 apply in the same way in both cases. In the analysis of the bilinear logarithmic terms in §5.1, however, our constructions apply in case **(i)** only under the additional assumption (5.2). The analysis of the field tensor terms in §5.2 was carried out only in case **(ii)** (and at present it is unclear how the results could be extended to case **(i)**). This gives a strong indication that the physically relevant scaling should indeed be described by case **(ii)**. This scaling can be realized by choosing the parameter p_{reg} in (3.33) sufficiently small. Thus in a physical model, the the parameter τ_{reg} in (2.43) should be chosen sufficiently large.

Arranging in this way that we are in case **(ii)**, it remains to justify the transition from the EL equations (4.5) to the stronger conditions (4.11). We already indicated an argument before Lemma 4.2. We are now in the position to make this argument precise: Recall that in case **(ii)**, the spectral projectors I_n are isospin-diagonal (3.43). The perturbation of these spectral projectors by the gauge phases leads to a finite hierarchy of equations to be satisfied in a weak evaluation on the light cone. With this in mind, it suffices to satisfy (4.5) with I_n according to (3.43). But clearly, we must take into account that the gauge phases enter the matrices $\mathcal{K}_{nc}$, as we now explain. We begin with the Dirac current terms. As the left-handed component of a wave function is modified by the gauge phases in the obvious way by

$$\chi_L\, \Psi(y) \to \chi_L\, \exp\left(-i \int_x^y A_L^j \xi_j\right) \Psi(y)\,,$$

the gauge potential enters the Dirac current term as described by the replacement

$$J_L \to \left(1 - i A_L^j\Big(\frac{x+y}{2}\Big)\, \xi_j\right) J_L\,.$$

In this way, the off-diagonal components of the Dirac current enter the diagonal matrix entries of $\mathcal{K}_n$ and thus the EL equations (4.5). Since the gauge currents have the same behavior under gauge transformations (see (5.3)), their off-diagonal elements enter the EL equations in the same way. For the mass terms, there is the complication that they have a *different* behavior under gauge transformations (for the logarithmic terms, this

was studied in (5.16), whereas for the contributions of the second order perturbation calculation, the dependence on the gauge phases can be read off from the formulas given in [4, Def. 7.2.1]). This different behavior under gauge transformation does not cause problems for the logarithmic poles, because we saw in §5.1 that the logarithmic poles on the light cone can be arranged to vanish. Thus the only effect of the different gauge behavior of the mass terms is that it modifies the values of the bosonic mass corresponding to the off-diagonal gauge potentials. The easiest method to describe this effect quantitatively is to again work with the EL equations (4.11), but to modify the off-diagonal matrix elements of $\mathcal{K}_L$ and $\mathcal{K}_R$ by multiplying the contributions (4.25)–(4.31) with numerical factors which take into account the linear behavior under off-diagonal left-handed gauge transformations. It is planned to work out the masses and coupling constants for a specific example of an admissible regularization in a separate publication.

8. The Einstein Equations

We come to the computation of the coupling constant in the Einstein-Hilbert action (6.21). We begin by computing the mapping Φ_T as defined by (6.17). Similar to (7.3), a short calculation using Lemma 4.11 gives

$$\Phi_T(\hat{T}_L, \hat{T}_R) = \Big((\hat{T}_R, \hat{T}_L) - \frac{1}{4}\, \mathrm{Tr}_{\mathbb{C}^2}\big(\hat{T}_L + \hat{T}_R\big)\, (\mathbb{1}, \mathbb{1}) \Big)\, K_8 + (\deg < 4)\,.$$

This mapping vanishes on $\mathfrak{J}_{\mathrm{curv}}$, so that condition (d) in §6.2 is violated. The way around this problem is to compute the next higher order in τ_{reg}:

Lemma 8.1. *The matrix Φ_T defined by* (6.17) *has the representation*

$$\begin{aligned}
\Phi_T(\hat{T}_L, \hat{T}_R) &= \Big((\hat{T}_R, \hat{T}_L) - \frac{1}{4}\, \mathrm{Tr}_{\mathbb{C}^2}\big(\hat{T}_L + \hat{T}_R\big)\, (\mathbb{1}, \mathbb{1}) \Big)\, K_8\, \Big(1 + \mathcal{O}\big((m\varepsilon)^{p_{reg}}\big) \Big) \\
&\quad + \tau_{reg}\, \mathrm{Tr}_{\mathbb{C}^2}\big(\hat{T}_L + \hat{T}_R\big) \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \right) K_{11} \\
&\quad + \mathcal{O}\big((m\varepsilon)^{2p_{reg}}\big)\, (\deg = 4) + (\deg < 4)\,,
\end{aligned}$$

where K_{11} is the following simple fraction of degree four,

$$K_{11} = \frac{1}{16\pi}\, \frac{1}{\overline{T^{(0)}_{[0]}}} \Big[T^{(-1)}_{[R,0]} T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} + c.c. \Big]\,. \tag{8.1}$$

Proof. Follows by a straightforward calculation similar as explained in the proof of Lemma 4.4. ◻

To explain the structure of the above result, we point out that the matrix Φ_T is trace-free. This can be understood immediately from the fact that (4.3) vanishes if the terms $\Delta|\lambda_{ncs}|$ are all equal.

Combining Lemma 4.12 with Lemma 4.2 and comparing with (6.17), one sees that

$$\Phi_{\mathrm{curv}}(1) = \frac{\tau_{\mathrm{reg}}}{2\delta^2} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) K_{16}\,.$$

A straightforward calculation shows that $\Phi_T|_{\mathfrak{J}_{\text{curv}}}$ as computed in Lemma 8.1 has the properties (d) and (e) in §6.2. Moreover, a short calculation yields that the condition (f) is also satisfied, and that

$$\Xi_T \circ \Phi_{\text{curv}}(1) = \frac{K_{16}}{\delta^2\, K_{11}}\, (\mathbb{1}, \mathbb{1})\,.$$

Taking the fermion-boson pairing with $(\mathbb{1}, \mathbb{1})$ and comparing with (6.21), we obtain the following result.

Theorem 8.2. *Assume that the parameters δ and p_{reg} satisfy the scaling* (4.87)*. Then effective action involves the Einstein-Hilbert part*

$$\mathcal{L}_{\text{EH}} = \frac{1}{\kappa}\, (R + 2\Lambda) + R \cdot \mathcal{O}\big(m^2\, (m\varepsilon)^{-p_{reg}}\big)\,, \tag{8.2}$$

where R is scalar curvature, κ is the gravitational constant and Λ the cosmological constant. The gravitational constant is given by

$$\kappa = \delta^2\, \frac{K_{11}}{K_{16}}\,,$$

where K_{11} and K_{16} are the simple fractions (8.1) *and* (4.90)*, both evaluated weakly on the light cone* (2.32)*.*

We point out that the cosmological constant remains undetermined; see the explanation after (6.21). But our result excludes corrections to the Einstein-Hilbert action of higher order in the curvature tensor. Note that the simple fractions K_{11} and K_{16} are both of degree four, and thus their quotient is of the order one. Hence

$$\kappa \sim \delta^2\,.$$

This means that the Planck length is to be identified with the length scale δ describing the shear and general surface states (see (2.34) and (2.39)).

We next explain how this theorem could be extended to the case

$$\delta \simeq \frac{1}{m}\, (m\varepsilon)^{\frac{p_{\text{reg}}}{2}}\,. \tag{8.3}$$

In this case, the terms $\sim m^2 R_{jk} \xi^j \xi^k$ in Lemma 4.12 are of the same order as those $\sim \tau_{\text{reg}}/\delta^2 R_{jk} \xi^j \xi^k$ and must be taken into account. They can be obtained by a straightforward computation. The statement of Theorem 8.2 will remain the same, except that the form of K_{16} will of course be modified, and that the error term in (8.2) disappears. The only structural difference is that (4.90) will then involve factors $T^{(1)}_{[0]}$, which have logarithmic poles on the light cone. It does not seem possible to compensate these logarithmic poles by a microlocal transformation. Therefore, in order for the logarithmic poles to drop out of the EL equations, one must impose that

$$\sum_{\alpha=1}^3 m_\alpha^2 = \sum_{\alpha=1}^3 \tilde{m}_\alpha^2\,.$$

This constraint for the neutrino masses can be understood similar as in Remark 4.9. Working out the detailed computations seems an interesting project for the future.

We finally note that for completeness, one should also compute how the energy-momentum tensor of the gauge fields enters the fermionic projector and verify that its effect on the EL equations to degree four is compatible with its coupling to the Einstein equations as given by the effective action. Since these computations are rather lengthy,

we postpone them to a future publication. These computations would conclude the complete analysis of the EL equations to degree four on the light cone up to errors of the order

$$Q(x,y) = (\deg = 4) \cdot o\big(|\vec{\xi}|^{-2}\big) + (\deg < 4)\,.$$

Appendix A. The Regularized Causal Perturbation Theory with Neutrinos

A.1. **The General Setting.** For clarity, we begin with a single Dirac sea (i.e. with one direct summand of (2.42) or (2.55)). Thus without regularization, the vacuum is described as the product of the Fourier integral (1.7) with a chiral asymmetry matrix,

$$P = Xt \qquad \text{with } t = P_m \text{ and } X = \mathbb{1}, \chi_L \text{ or } \chi_R\,, \tag{A.1}$$

under the constraint that $X = \mathbb{1}$ if $m > 0$. We again denote the regularization by an index ε. We always assume that the regularization is *homogeneous*, so that P^ε is a multiplication operator in momentum space, which sometimes we denote for clarity by $\hat{P}^\varepsilon$. If $m > 0$, we assume that the regularization satisfies all the conditions in [4, Chapter 4]; see also the compilation in [7, Section 3]. In the case $m = 0$, we relax the conditions on the shear and allow for general surface states, as explained in Section 2.3. In the low-energy regime, P^ε should still be of the form (A.1), i.e.

$$\hat{P}^\varepsilon(k) = \begin{cases} (\not{k} + m)\, \delta\big(k^2 - m^2\big) & \text{if } m > 0 \\ X\, \not{k}\, \delta(k^2) & \text{if } m = 0 \end{cases} \qquad \big(|k^0| + |\vec{k}| \ll \varepsilon^{-1}\big)\,. \tag{A.2}$$

However, in the high-energy regime, P^ε will no longer satisfy the Dirac equation. But in preparation of the perturbation expansion, we need to associate the states P^ε to eigenstates of the Dirac operator (not necessarily to the eigenvalue m). To this end, we introduce two operators V_{shift} and V_{shear} with the following properties. The operator V_{shift} has the purpose of changing the momentum of states such that general surface states (as in Figure 1 (B)) are mapped onto the mass cone, i.e.

$$\big(V_{\text{shift}} \psi\big)(k) := \psi\big(v_{\text{shift}}(k)\big)\,, \tag{A.3}$$

where $v_{\text{shift}} : \hat{M} \to \hat{M}$ is a diffeomorphism. The operator V_{shear}, on the other hand, is a unitary multiplication operator in momentum space, which has the purpose of introducing the shear of the surface states (i.e. it should map the states in Figure 1 (B) to those in Figure 1 (C)),

$$\big(V_{\text{shear}} \psi\big)(k) = \hat{V}_{\text{shear}}(k)\, \psi(k) \qquad \text{with} \qquad \hat{V}_{\text{shear}}(k) \text{ unitary}\,.$$

These operators are to be chosen such that the operator $\check{P}^\varepsilon$ defined by

$$P^\varepsilon = V_{\text{shear}}\, V_{\text{shift}}\, \check{P}^\varepsilon\, V_{\text{shift}}^{-1}\, V_{\text{shear}}^{-1} \tag{A.4}$$

is of the following form,

$$\check{P}^\varepsilon(k) = \begin{cases} d(k)\Big(\not{k} + m(k)\,\mathbb{1}\Big)\delta\big(k^2 - m(k)^2\big) & \text{if } m > 0 \\ d(k)\, X\, \not{k}\, \delta(k^2) & \text{if } m = 0 \end{cases} \tag{A.5}$$

with X as in (A.1). Thus in the massive case, $\check{P}^\varepsilon$ should be composed of Dirac eigenstates corresponding to an energy-dependent mass $m(k) > 0$, and it should have vector-scalar structure. In the massless case, we demand that $k^2 = 0$, so that the states of $\check{P}^\varepsilon$ are all neutral. The ansatz (A.5) is partly a matter of convenience, and partly a requirement needed for the perturbation expansion (see Proposition A.1 below).

Moreover, we assume for convenience that $\check{P}^\varepsilon$ is only composed of states of negative energy,

$$\check{P}^\varepsilon(k) = 0 \qquad \text{if } k^2 < 0 \text{ or } k^0 > 0\,. \tag{A.6}$$

In view of (A.2), it is easiest to assume that V_{shift} and V_{shear} are the identity in the low-energy regime, i.e.

$$\hat{V}_{\text{shear}}(k) = \mathbb{1} \quad \text{and} \quad v_{\text{shift}}(k) = k \qquad \text{if } |k^0| + |\vec{k}| \ll \varepsilon^{-1}\,. \tag{A.7}$$

Then, by comparing (A.2) with (A.5), one finds that

$$d(k) = \mathbb{1} \quad \text{and} \quad m(k) = m \qquad \text{if } |k^0| + |\vec{k}| \ll \varepsilon^{-1}\,. \tag{A.8}$$

The required regularization of $P^\varepsilon(x,y)$ on the scale ε is implemented by demanding that

$$d(k) \text{ decays on the scale } |k^0| + |\vec{k}| \sim \varepsilon^{-1}\,. \tag{A.9}$$

In view of their behavior in the low-energy regime, it is natural to assume that the functions in (A.7) and (A.8) should be smooth in momentum space and that their derivatives scale in powers of the regularization length, i.e.

$$\begin{aligned} \big|\nabla^\gamma_k d(k)\big| &\sim \varepsilon^{|\gamma|} |d(k)|\,, & \big|\nabla^\gamma_k m(k)\big| &\sim \varepsilon^{|\gamma|} |m(k)| \\ \big|\nabla^\gamma_k \hat{V}_{\text{shear}}(k)\big| &\sim \varepsilon^{|\gamma|} |\hat{V}_{\text{shear}}(k)|\,, & \big|\nabla^\gamma_k v_{\text{shift}}(k)\big| &\sim \varepsilon^{|\gamma|} |v_{\text{shift}}(k)|\,. \end{aligned} \tag{A.10}$$

Clearly, the above conditions do not uniquely determine the function d and the operators V_{shift} and V_{shear}. But we shall see that the results of our analysis will be independent of the choice of these operators. We remark that the transformation V_{shear} is analogous to the transformations U_l considered in [4, Appendix D] (see [4, eq. (D.22)]), except that here we consider only one unitary transformation.

The last construction immediately generalizes to a system of Dirac seas. Namely, suppose that without regularization, the auxiliary fermionic projector of the vacuum is a direct sum of Dirac seas (see for example (2.42) or (2.55)),

$$P^{\text{aux}} = \bigoplus_{\ell=1}^{\ell_{\max}} X_\ell\, t_\ell\,.$$

Then we introduce P^ε simply by taking the direct sum of the corresponding regularized seas

$$P^{\text{aux}} := \bigoplus_{\ell=1}^{\ell_{\max}} P^\varepsilon_\ell\,, \qquad V_{\text{shift}} := \bigoplus_{\ell=1}^{\ell_{\max}} V^\ell_{\text{shift}}\,, \qquad V_{\text{shear}} := \bigoplus_{\ell=1}^{\ell_{\max}} V^\ell_{\text{shear}}\,.$$

Setting

$$P^{\text{aux}} = V_{\text{shear}}\, V_{\text{shift}}\, \check{P}^\varepsilon\, V_{\text{shift}}^{-1}\, V_{\text{shear}}^{-1}\,,$$

the operator $\check{P}^\varepsilon$ satisfies the Dirac equation

$$(\not{k} - mY(k))\, \hat{P}^\varepsilon(k) = 0\,, \tag{A.11}$$

where the mass matrix is given by (cf. (2.44) or (2.57))

$$mY(k) = \bigoplus_{\ell=1}^{\ell_{\max}} m_\ell\,.$$

In the low-energy regime, we know furthermore that

$$\left. \begin{array}{c} \hat{V}_{\text{shear}}(k) = \mathbb{1} \quad \text{and} \quad v_{\text{shift}}(k) = k \\ \hat{P}^{\text{aux}}(k) = X\,t \end{array} \right\} \qquad \text{if } |k^0| + |\vec{k}| \ll \varepsilon^{-1} \,,$$

where X and t are given as in (2.45). Clearly, the regularity assumptions (A.10) are imposed similarly to $\hat{P}^{\text{aux}}$. Finally, we need to specify what we mean by saying that two Dirac seas are regularized in the same way. The difficulty is that, as mentioned above, different choices of d, $\hat{V}_{\text{shear}}$ and $\hat{V}_{\text{shift}}$ may give rise to the same regularization effects. In order to keep the situation reasonably simple, we use the convention that if we want two Dirac seas to show the same regularization effects, we choose the corresponding functions d as well as $\hat{V}_{\text{shear}}$ and $\hat{V}_{\text{shift}}$ to be exactly the same. If conversely two Dirac seas should show different regularization effects, we already choose the corresponding functions d to be different. Then we can say that two Dirac seas labeled by a and b are *regularized in the same way* if $d_a \equiv d_b$. In this case, our convention is that also $(\hat{V}_{\text{shear}})_a = (\hat{V}_{\text{shear}})_b$ and $(\hat{V}_{\text{shift}})_a = (\hat{V}_{\text{shift}})_b$. This notion gives rise to an equivalence relation on the Dirac seas. In the formalism of Section 2.6, the equivalence classes will be labeled by the parameters τ_i^{reg} (see (2.43) and (2.58)).

A.2. **Formal Introduction of the Interaction.** Now the interaction can be introduced most conveniently by using the *unitary perturbation flow* [10, Section 5]. In order not to get confused with the mass matrix, we introduce an additional spectral parameter μ into the free Dirac equation, which in momentum space reads

$$(\not{k} - mY(k) - \mu\mathbb{1})\hat{\Psi}(k) = 0\,.$$

For this Dirac equation, we can introduce the spectral projectors p, the causal fundamental solutions k and the symmetric Green's functions s can be introduced just as in [4, §2.2], if only in the formulas in momentum space we replace m by $mY(k)$. For clarity, we denote the dependence on μ by an subscript $+\mu$ (this notation was used similarly in [4, §2.6]; see also [4, §C.3] for an additional "modified mass scaling", which we will for simplicity not consider here). We describe the interaction by inserting an operator $\mathcal{B}$ into the Dirac operator,

$$\mathcal{D} = i\not{\partial} + \mathcal{B} - mY(k)\,.$$

After adding the subscript $+\mu$ to all factors p, k or s in the operator products in [10, Section 5], we obtain an operator U which associates to every solution ψ of the free Dirac equation $(i\not{\partial} - mY - \mu\,\mathbb{1})\psi = 0$ a corresponding solution $\tilde{\psi}$ of the interacting Dirac equation $(i\not{\partial} + \mathcal{B} - mY - \mu\,\mathbb{1})\psi = 0$,

$$U(\mathcal{B})\;:\;\psi \mapsto \tilde{\psi}\,.$$

The operator U is uniquely defined in terms of a formal power series in $\mathcal{B}$. Taking μ as a free parameter, in [10, Section 5] the operator U is shown to be unitary with respect to the indefinite inner product (2.3). We now use U to unitarily transform all the Dirac states contained in the operator $\check{P}^\varepsilon$ and set

$$\boxed{\tilde{P}^{\text{aux}} = V_{\text{shear}}\,V_{\text{shift}}\,U(\mathcal{B})\,\check{P}^\varepsilon\,U(\mathcal{B})^{-1}\,V_{\text{shift}}^{-1}\,V_{\text{shear}}^{-1}\,.} \tag{A.12}$$

This construction uniquely defines the regularized auxiliary fermionic projector with interaction $\tilde{P}^{\text{aux}}$ in terms of a formal power expansion in $\mathcal{B}$. The fermionic projector is then obtained by taking the partial trace (see (2.53) or (2.56)).

A.3. **Compatibility Conditions for the Interaction.** In order to derive the structure of the admissible $\mathcal{B}$, we first consider a perturbation calculation to first order and assume that $\mathcal{B}$ is a multiplication operator in position space having the form of a plane wave of momentum q,

$$\mathcal{B}(x) = \mathcal{B}_q \, e^{-iqx} \, . \tag{A.13}$$

In this case (cf. [4, eq. (D.14)]),

$$\Delta \check{P}^{\text{aux}} = -\int_{-\infty}^{\infty} d\mu \left(s_{+\mu} \, \mathcal{B} \, p_{+\mu} \, \check{P}^\varepsilon \;+\; \check{P}^\varepsilon \, p_{+\mu} \, \mathcal{B} \, s_{+\mu} \right) \, .$$

Using a matrix notation in the direct sums with indices $a, b \in \{1, \ldots, \ell_{\max}\}$, we obtain in momentum space (for the notation see [4, Chapter 2] or [10])

$$\begin{aligned} (\Delta \check{P}^\varepsilon)^a_b(k+q, k) = -\int_{-\infty}^{\infty} d\mu \Big\{ & s_{m_a+\mu}(k+q) \, (\mathcal{B}_q)^a_b \, p_{m_b+\mu}(k) \, (\check{P}^\varepsilon)^b_b(k) \\ & +(\check{P}^\varepsilon)^a_a(k+q) \, p_{m_a+\mu}(k+q) \, (\mathcal{B}_q)^a_b \, s_{m_b+\mu}(k) \Big) \Big\} \, . \end{aligned} \tag{A.14}$$

This equation was already considered in [2, Section 3] and [4, Appendix D]. However, here we analyze the situation more systematically and in a more general context, pointing out the partial results which were obtained previously.

For clarity, we analyze (A.14) step by step, beginning with the diagonal elements. For ease in notation, we assume that $\mathcal{B}_q$ has only one non-trivial component, which is on the diagonal,

$$(\mathcal{B}_q)^a_b = \delta^{a\ell} \, \delta_{b\ell} \, \mathcal{B} \tag{A.15}$$

with $\ell \in \{1, \ldots, \ell_{\max}\}$ and $\mathcal{B}$ a matrix acting on Dirac spinors. Shifting the integration variable according to $m_\ell + \mu \to \mu$, we obtain (cf. [4, eq. (D.15)])

$$\begin{aligned} &(\Delta \check{P}^\varepsilon)^\ell_\ell(k+q, k) \\ &= -\int_{-\infty}^{\infty} d\mu \Big\{ s_\mu(k+q) \, \mathcal{B} \, p_\mu(k) \, (\check{P}^\varepsilon)^\ell_\ell(k) + (\check{P}^\varepsilon)^\ell_\ell(k+q) \, p_\mu(k+q) \, \mathcal{B} \, s_\mu(k) \Big\} \\ &= -\int_{-\infty}^{\infty} d\mu \, \epsilon(\mu) \left\{ \frac{\text{PP}}{(k+q)^2 - \mu^2} \, (\not{k} + \not{q} + \mu) \, \mathcal{B} \, (\not{k} + \mu) \, \delta(k^2 - \mu^2) \, (\check{P}^\varepsilon)^\ell_\ell(k) \right. \\ &\qquad\qquad \left. + \, (\check{P}^\varepsilon)^\ell_\ell(k+q) \, \delta((k+q)^2 - \mu^2) \, (\not{k} + \not{q} + \mu) \, \mathcal{B} \, (\not{k} + \mu) \, \frac{\text{PP}}{k^2 - \mu^2} \right\} \\ &= -\int_{-\infty}^{\infty} d\mu \, \epsilon(\mu) \, \frac{\text{PP}}{2kq + q^2} \, \Big\{ (\not{k} + \not{q} + \mu) \, \mathcal{B} \, (\not{k} + \mu) \, \delta(k^2 - \mu^2) \, (\check{P}^\varepsilon)^\ell_\ell(k) \\ &\qquad\qquad - \, (\check{P}^\varepsilon)^\ell_\ell(k+q) \, \delta((k+q)^2 - \mu^2) \, (\not{k} + \not{q} + \mu) \, \mathcal{B} \, (\not{k} + \mu) \Big\}, \end{aligned}$$

where in the last step we used that the argument of the δ-distribution vanishes. Carrying out the μ-integration gives (cf. [4, eq. (D.15)])

$$\begin{aligned} (\Delta \check{P}^\varepsilon)^\ell_\ell(k+q, k) = -\frac{\text{PP}}{4kq + 2q^2} \, \Big\{ & \Big((\not{k} + \not{q}) \, \mathcal{B} + \mathcal{B} \, \not{k} \Big) \, (\check{P}^\varepsilon)^\ell_\ell(k) \\ & -(\check{P}^\varepsilon)^\ell_\ell(k+q) \Big((\not{k} + \not{q}) \, \mathcal{B} + \mathcal{B} \, \not{k} \Big) \Big\} \, . \end{aligned} \tag{A.16}$$

Here the principal part has poles if $2kq + q^2 = 0$, leading to a potential divergence of $\Delta \check{P}^\varepsilon$. In order to explain the nature of this divergence, we first point out that if $\mathcal{B}$ had been chosen to be a smooth function with rapid decay, then $\Delta \check{P}$ would have been finite (see the proof of [4, Lemma 2.2.2]). Thus the potential divergence is related to the

fact that the plane wave in (A.13) does *not* decay at infinity. A more detailed picture is obtained by performing the light-cone expansion (see [2] and [4, Appendix F]). Then one can introduce the notion that (A.16) is causal if its light-cone expansion only involves integrals along a line segment $\overline{xy}$. Since such integrals are uniformly bounded, it follows immediately that all contributions to the light-cone expansion are finite for all q. If conversely (A.16) diverges, then the analysis in [4, Appendix F] reveals that individual contributions to the light-cone expansion do diverge, so that unbounded line integrals must appear (see also the explicit light-cone expansions in [1]). In this way, one gets a connection between the boundedness of (A.16) and the *causality of the light-cone expansion*.

Unbounded line integrals lead to contributions to the EL equations whose scaling behavior in the radius is different from all other contributions. Therefore, the EL equations are satisfied only if all unbounded line integrals drop out. The easiest way to arrange this is to demand that the fermionic projector itself should not involve any unbounded line integrals. This is our motivation for imposing that

$$(\Delta \hat{P})^\varepsilon(k+q,k) \text{ should be bounded locally uniformly in } q. \tag{A.17}$$

Let us analyze this boundedness condition for (A.16). Since the denominator in (A.16) vanishes at $q \to 0$, we clearly get the necessary condition that the curly brackets must vanish at $q=0$,

$$\big[\{ \not{k}, \mathcal{B} \} \,,\, \check{P}^\varepsilon(k) \big] = 0 \,. \tag{A.18}$$

Using (A.5) together with the identity

$$[\{\not{k}, \mathcal{B}\} \,,\, \not{k}] = \big[k^2, \mathcal{B}\big] + \not{k}\mathcal{B}\not{k} - \not{k}\mathcal{B}\not{k} = 0 \,,$$

we find that (A.18) is automatically satisfied in the case $X = \mathbb{1}$. The situation is more interesting if a chiral asymmetry is present. If for example $X = \chi_L$, we get the condition

$$[\{\not{k}, \mathcal{B}\} \,,\, \chi_L \not{k}] = 0 \,.$$

This condition is again trivial if $\mathcal{B}$ is odd (meaning that $\{\mathcal{B}, \gamma^5\} = 0$). However, if $\mathcal{B}$ is even, we conclude that

$$0 = \frac{\gamma^5}{2} \, \{\{\not{k}, \mathcal{B}\} \,,\, \not{k}\} = \gamma^5 \, \not{k} \, \mathcal{B} \, \not{k}$$

(where in the last step we used that $k^2 = 0$ in view of (A.5)). As k is any state on the lower mass shell, this rules out that $\mathcal{B}$ is a bilinear potential, leaving us with a scalar or a pseudoscalar potential. In order to rule out these potentials, we next choose a vector $\hat{q}$ with $\hat{q}k = 0$, set $q = \varepsilon \hat{q}$ and consider (A.16) in the limit $\varepsilon \to 0$. Then the denominator in (A.16) diverges like ε^{-2}, so that the curly brackets must tend to zero even $\sim \varepsilon^2$.

$$\Big((\not{k} + \varepsilon \not{\hat{q}})\mathcal{B} + \mathcal{B}\not{k} \Big) \check{P}^\varepsilon(k) - \check{P}^\varepsilon(k+\varepsilon\hat{q}) \Big((\not{k} + \varepsilon \not{\hat{q}})\mathcal{B} + \mathcal{B}\not{k} \Big) = \mathcal{O}(\varepsilon^2) \,. \tag{A.19}$$

Using that $\check{P}^\varepsilon(k)$ is left-handed and that $\mathcal{B}$ is even, we find that the first summand in (A.19) is right-handed, whereas the second summand is left-handed. Hence both summand must vanish separately, and thus

$$0 = \Big((\not{k} + \not{\hat{q}})\mathcal{B} + \mathcal{B}\not{k} \Big) \check{P}^\varepsilon(k) = \not{\hat{q}} \, \mathcal{B} \, \not{k} \, d(k) \, \delta(k^2) \,.$$

This condition implies that $\mathcal{B}$ must vanish. We conclude that if $X = \chi_L$, only odd potentials may occur. We can write this result more generally as

$$\boxed{\mathcal{B}\, X = X^* \,\mathcal{B}\,.} \tag{A.20}$$

We have thus derived the *causality compatibility condition* (2.49) from our boundedness condition (A.17). This derivation is an alternative to the method in [4, §2.3], where the same condition was introduced by the requirement that it should be possible to commute the chiral asymmetry matrix through the perturbation expansion.

So far, we considered (A.17) in the limit $q \to 0$. We now analyze this condition for general q. Using (A.5) and (A.20), a short calculation gives

$$(\Delta \check{P}^\varepsilon)^\ell_\ell(k+q,k) = -X\, \frac{(\not{k}+\not{q})\,\mathcal{B}\,\not{k}}{4kq+2q^2} \Big(d(k)\,\delta(k^2-m^2) - d(k+q)\,\delta((k+q)^2-m^2)\Big),$$

we we set $m = m_\ell$. If $d(k) = d(k+q)$, the transformations

$$\begin{aligned}
\int_0^1 \delta'\big(k^2-m^2+\tau(2kq+q^2)\big)\, d\tau &= \frac{1}{2kq+q^2} \int_0^1 \frac{d}{d\tau} \delta\big(k^2-m^2+\tau(2kq+q^2)\big)\, d\tau \\
&= \frac{1}{2kq+q^2} \Big(\delta((k+q)^2-m^2) - \delta(k^2-m^2) \Big)
\end{aligned}$$

show that $\Delta \check{P}^\varepsilon$ is indeed a bounded distribution for any q. Thus it remains to be concerned about the contribution if $\delta(k) \neq \delta(k+q)$,

$$X\, \frac{(\not{k}+\not{q})\,\mathcal{B}\,\not{k}}{4kq+2q^2}\, \delta(k^2-m^2)\, \Big(d(k+q) - d(k) \Big)\,. \tag{A.21}$$

Unless in the trivial case $\mathcal{B} = 0$, this contribution is infinite at the poles of the denominator. We conclude that in order to comply with the condition (A.17), we must impose that the weight function $d(k)$ in (A.5) is constant on the mass shell $k^2 = m(k)^2$. This is indeed the case in the low-energy regime (A.8). However, in the high-energy region, the function $d(k)$ is in general not a constant (and indeed, assuming that $d(k)$ is constant would be in contradiction to (A.9)). Our way out of this problem is to observe that (A.21) implies that the light-cone expansion of (A.16) is in general not causal, in the sense that it involves unbounded line integrals. However, using that $d(k+q) - d(k) \sim q|\nabla d|$, the scalings $q \sim \ell_{\text{macro}}^{-1}$ and (A.10) show that these non-causal contributions to the light-cone expansion are of

$$\text{higher order in } \varepsilon/\ell_{\text{macro}}\,. \tag{A.22}$$

This consideration shows that the perturbation expansion will give rise to error terms of higher order in $\varepsilon/\ell_{\text{macro}}$. In what follows, we will always neglect such error terms. If this is done, the above assumptions are consistent and in agreement with (A.17), provided that the causality compatibility condition (A.20) holds.

Before moving on to potentials which mix different Dirac seas, we remark that the above arguments can also be used to derive constraints for the possible form of $\hat{P}^\varepsilon(k)$, thus partly justifying our ansatz A.5.

Proposition A.1 (Possible form of $\check{P}^\varepsilon$)**.** *Suppose that $\Delta \check{P}^\varepsilon$ as given by* (A.16) *satisfies the condition* (A.17)*. We renounce the assumptions on $\check{P}^\varepsilon$ (see* (A.5)*,* (A.6)*,* (A.8) *and* (A.10)*), but we assume instead that the admissible interaction includes chiral or axial potentials. Then for every k, there are complex coefficients $a, b, c, d \in \mathbb{C}$ such that*

$$\check{P}^\varepsilon(k) = a\mathbb{1} + ib\gamma^5 + c\not{k} + d\gamma^5 \not{k}\,. \tag{A.23}$$

Proof. We again analyze the necessary condition (A.18). This condition is clearly satisfied if $\mathcal{B}$ is a vector potential. Thus by linearity, we may assume that $\mathcal{B} = \gamma^5 \slashed{A}$ is a vector potential. It follows that

$$0 = \left[\left\{ \slashed{k}, \gamma^5 \slashed{A} \right\}, \check{P}^\varepsilon(k) \right] = \left[\gamma^5 \left[\slashed{A}, \slashed{k} \right], \check{P}^\varepsilon(k) \right] .$$

Decomposing $\check{P}^\varepsilon(k)$ into its even and odd components, by linearity we can again consider these components after each other. If $\check{P}^\varepsilon(k)$ is odd, i.e.

$$\check{P}^\varepsilon(k) = \slashed{u} + \gamma^5 \slashed{v} ,$$

we obtain

$$0 = \gamma^5 \left\{ \left[\slashed{A}, \slashed{k} \right], \check{P}^\varepsilon(k) \right\} = 4\gamma^5 \big((ku)\, \slashed{A} - (Au)\, \slashed{k} \big) + 4\big((kv)\, \slashed{A} - (Av)\, \slashed{k} \big) .$$

Since A is arbitrary, it follows that u and v must be multiples of k.

If $\check{P}^\varepsilon(k)$ is even, we obtain the condition

$$\left[\left[\slashed{A}, \slashed{k} \right], \check{P}^\varepsilon(k) \right] = 0 .$$

This condition is obviously satisfied if $\check{P}^\varepsilon(k)$ is a scalar or a pseudoscalar. Writing the bilinear component of $\check{P}^\varepsilon(k)$ in the form $F_{ij}\gamma^i\gamma^j$ with an anti-symmetric tensor field F, we get the condition

$$0 = \left[\left[\slashed{A}, \slashed{k} \right], F_{ij}\gamma^i\gamma^j \right] = 2F_{ij}k^i \left[\slashed{A}, \gamma^j \right] - 2F_{ij}A^i \left[\slashed{k}, \gamma^j \right] .$$

Since A is arbitrary, it follows that $F = 0$, concluding the proof. □

The step from (A.23) to our the stronger assumption (A.2) could be justified by the assumption that the image of P^ε should be negative definite or neutral, and furthermore by assuming that without chiral asymmetry the matrix γ^5 is absent, whereas the chiral asymmetry is then introduced simply by multiplying with χ_L or χ_R. We finally remark that in [4, Appendix D], the condition (A.18) is analyzed for $\mathcal{B}$ a scalar potential to conclude that $\check{P}^\varepsilon(k)$ should commute with the Dirac operator (see [4, eq. (D.16) and eq. (D.17)]). This is consistent with our ansatz (A.5), which is even a solution of the Dirac equation (A.11). However, we here preferred to avoid working with scalar potentials, which do not seem crucial for physically realistic models.

We next consider instead of (A.15) a general potential $\mathcal{B}_q$ which may have off-diagonal terms in the direct summands. Then (A.14) can be evaluated similar as in the computation after (A.15), but the calculation is a bit more complicated. Therefore, we first compute the integral of the first summand in (A.14),

$$\begin{aligned}
&\int_{-\infty}^\infty d\mu\; s_{m_a+\mu}(k+q)\; (\mathcal{B}_q)^a_b\; p_{m_b+\mu}(k)\; (\check{P}^\varepsilon)^b_b(k) \\
&\quad = \int_{-\infty}^\infty d\mu\; \epsilon(m_b+\mu)\; \frac{\mathrm{PP}}{(k+q)^2 - (m_a+\mu)^2}\; \delta(k^2 - (m_b+\mu)^2) \\
&\qquad\qquad\qquad \times (\slashed{k} + \slashed{q} + m_a + \mu)\; (\mathcal{B}_q)^a_b\; (\slashed{k} + m_b + \mu)\; (\check{P}^\varepsilon)^b_b(k) \\
&\quad = \sum_{\mu = \pm|k| - m_b} \frac{1}{m_b + \mu}\; \mathfrak{B}^a_b\; (\check{P}^\varepsilon)^b_b(k) , \qquad\qquad (A.24)
\end{aligned}$$

where we set

$$\mathfrak{B}^a_b = \frac{1}{2}\, \mathrm{PP}\bigg(\frac{(\slashed{k} + \slashed{q} + m_a + \mu)\; (\mathcal{B}_q)^a_b\; (\slashed{k} + m_b + \mu)}{2kq + q^2 - (m_a^2 - m_b^2) - 2\mu(m_a - m_b)} \bigg) \qquad (A.25)$$

and $|k| = \sqrt{k^2}$ (note that, in view of our assumption (A.6), the factor $(\check{P}^\varepsilon)^b_b(k)$ guarantees that the above expression vanishes if $k^2 < 0$). Treating the second summand in (A.14) similarly, we obtain

$$(\Delta \check{P}^\varepsilon)^a_b(k+q,k) = \sum_{\mu=\pm|k+q|-m_a} \frac{(\check{P}^\varepsilon)^a_a(k+q)}{m_a+\mu} \; \mathfrak{B}^a_b \; - \sum_{\mu=\pm|k|-m_b} \mathfrak{B}^a_b \, \frac{(\check{P}^\varepsilon)^b_b(k)}{m_b+\mu} \,. \quad \text{(A.26)}$$

This formula is rather involved, but fortunately we do not need to enter a detailed analysis. It suffices to observe that (A.25) has poles in q, which lead to singularities of (A.24). Thus the only way to satisfy the condition (A.17) is to arrange that contributions of the first and second expression on the right of (A.26) cancel each other. In view of (A.5), the first expression involves $d_a(k+q)$, whereas in the second expression the term $d_b(k)$ appears. This shows that in order to get the required cancellations, the functions d_a and d_b must coincide. Using the notion introduced on page 77, we conclude that $\mathcal{B}$ may describe an interaction of Dirac seas only if they are regularized in the same way. An interaction of Dirac seas with different regularization, however, is prohibited by the causality condition for the light-cone expansion. For brevity, we also say that the interaction must be *regularity compatible*.

A.4. **The Causal Perturbation Expansion with Regularization.** We are now ready to perform the causal perturbation expansion. In [10, Section 5], the unitary perturbation flow is introduced in terms of an operator product expansion. Replacing the Green's functions and fundamental solutions in this expansion by the corresponding operators of the free Dirac equation (A.11), we can write the operator $U(\mathcal{B})\, \check{P}^\varepsilon\, U(\mathcal{B})^{-1}$ as a series of operator products of the form

$$\mathcal{Z} := C_1 \,\mathcal{B}\, \cdots \,\mathcal{B}\, C_p \,\mathcal{B}\, \check{P}^\varepsilon \,\mathcal{B}\, C_{p+1} \,\mathcal{B}\, \cdots \,\mathcal{B}\, C_k \,,$$

where the factors C_l are the Green's functions or fundamental solutions of the free Dirac equation (A.11). The operators C_l are diagonal in momentum space, whereas the potential $\mathcal{B}$ varies on the macroscopic scale and thus changes the momentum only on the scale ℓ_{macro}^{-1}. Thus all the factors C_l will be evaluated at the same momentum p, up to errors of the order ℓ_{macro}^{-1}. We refer to this momentum p, determined only up to summands of the order ℓ_{macro}^{-1} as the considered *momentum scale*. In view of the regularity assumptions on the functions d and m in (A.10), we may replace them by the constants $d(p)$ and $m(p)$, making an error of the order (A.22). This evaluation of the regularization functions is referred to as the *fixing of the momentum scale*, and we indicate it symbolically by $|_{\text{scale } p}$. Since $\mathcal{B}$ is regularity compatible, we may then commute the constant matrix d to the left. Moreover, we can apply the causality compatibility condition (A.20) together with the form of X in (A.5) and (A.2) to also commute the chiral asymmetry matrix X to the left. We thus obtain the expansion

$$U(\mathcal{B})\, \check{P}^\varepsilon\, U(\mathcal{B})^{-1}\big|_{\text{scale } p} = \sum_{k=0}^\infty \sum_{\alpha=0}^{\alpha_{\max}(k)} c_\alpha\, X d\, C_{1,\alpha} \,\mathcal{B}\, C_{2,\alpha} \,\mathcal{B}\, \cdots \,\mathcal{B}\, C_{k+1,\alpha}\big|_{\text{scale } p} \quad \text{(A.27)}$$
$$+ \,(\text{higher orders in } \varepsilon/\ell_{\text{macro}})\,,$$

where we set

$$X = \bigoplus_{\ell=1}^{\ell_{\max}} X_\ell \qquad \text{and} \qquad d = \bigoplus_{\ell=1}^{\ell_{\max}} d_\ell(p)\,,$$

and c_α are combinatorial factors. Here the combinatorics of the operator products coincides precisely with that of the causal perturbation expansion for the fermionic projector in [10, Theorem 4.1].

A.5. **The Behavior under Gauge Transformations.** In order to analyze the behavior of the above expansion under $U(1)$-gauge transformations, we consider the case of a pure gauge potential, i.e. $\mathcal{B} = \partial\!\!\!/\Lambda$ with a real-valued function Λ. Then the gauge invariance of the causal perturbation expansion yields

$$U(\mathcal{B})\, \check{P}^\varepsilon\, U(\mathcal{B})^{-1}\big|_{\text{scale } p} = (e^{i\Lambda} \check{P}^\varepsilon e^{-i\Lambda})\big|_{\text{scale } p} + (\text{higher orders in } \varepsilon/\ell_{\text{macro}})\,. \quad \text{(A.28)}$$

According to (A.12), we obtain $\tilde{P}^{\text{aux}}$ by applying with V_{shift} and V_{shear}. The transformation V_{shift} is a subtle point which requires a detailed explanation. We first consider its action on a multiplication operator in momentum space $M(k)$. Then, according to the definition (A.3),

$$\begin{aligned}(V_{\text{shift}}\, M\, V_{\text{shift}}^{-1}\psi)(k) &= (M V_{\text{shift}}^{-1}\psi)\big(v_{\text{shift}}(k)\big) \\ &= M\big(v_{\text{shift}}(k)\big)\, (V_{\text{shift}}^{-1}\psi)\big(v_{\text{shift}}(k)\big) = M\big(v_{\text{shift}}(k)\big)\, \psi(k) \quad \text{(A.29)}\end{aligned}$$

so that the transformation again yields a multiplication operator, but with a transformed argument. To derive the transformation law for multiplication operators in position space, we first let $f = e^{-iqx}$ be the operator of multiplication by a plane wave. Then

$$\begin{aligned}(V_{\text{shift}}\, f\, V_{\text{shift}}^{-1}\psi)(k) &= (f V_{\text{shift}}^{-1}\psi)\big(v_{\text{shift}}(k)\big) \\ &= (V_{\text{shift}}^{-1}\psi)\big(v_{\text{shift}}(k) - q\big) = \psi\Big(v_{\text{shift}}^{-1}\big(v_{\text{shift}}(k) - q\big)\Big).\end{aligned}$$

This can be simplified further if we assume that the momentum $q \sim \ell_{\text{macro}}^{-1}$ is macroscopic. Namely, the scaling of the function v_{shift} in (A.10) allows us to expand in a Taylor series in q,

$$(V_{\text{shift}}\, f\, V_{\text{shift}}^{-1}\psi)(k) = \psi\Big(k - Dv_{\text{shift}}^{-1}\big|_{v_{\text{shift}}(k)}\, q\Big) + (\text{higher orders in } \varepsilon/\ell_{\text{macro}})\,.$$

Thus $V_{\text{shift}}\, f\, V_{\text{shift}}^{-1}$ is again a multiplication operator in position space, but now corresponding to the new momentum

$$L(k)\, q \qquad \text{with} \qquad L(k) := Dv_{\text{shift}}^{-1}\big|_{v_{\text{shift}}(k)}\,.$$

Again in view of the regularity assumptions (A.10), when fixing the momentum scale we may replace the argument k by p, i.e.

$$V_{\text{shift}}\, e^{-iqx}\, V_{\text{shift}}^{-1}\big|_{\text{scale } p} = e^{-i(L(p)q)\,x} + (\text{higher orders in } \varepsilon/\ell_{\text{macro}})\,.$$

Using the relation $(L(p)q)\,x = q\,L(p)^*x$, we can rewrite this transformation law simply as a linear transformation of the space-time coordinates. Then the transformation law generalizes by linearity to a general multiplication operator by a function f which varies on the macroscopic scale, i.e.

$$V_{\text{shift}}\, f(x)\, V_{\text{shift}}^{-1}\big|_{\text{scale } p} = f\big(L(p)^*x\big) + (\text{higher orders in } \varepsilon/\ell_{\text{macro}})\,. \quad \text{(A.30)}$$

With (A.29) and (A.30), we can transform (A.28) to the required form (A.12). Using that V_{shear} commutes with scalar and macroscopic multiplication operators (again up to higher orders in $\varepsilon/\ell_{\text{macro}}$), we obtain in view of (A.4)

$$P^\varepsilon(x,y)\big|_{\text{scale p}} = e^{i\Lambda(L(p)^*x)}\, P^\varepsilon(x,y)\, e^{-i\Lambda(L(p)^*y)} + (\text{higher orders in } \varepsilon/\ell_{\text{macro}})\,. \quad \text{(A.31)}$$

Except for the factors $L(p)^*$, this formula describes the usual behavior of the fermionic projector under gauge transformations. In particular, if we do not consider general surface states and $V_{\text{shift}} = \mathbb{1}$, then our perturbation expansion is gauge invariant. However, if we consider general surface states described by a non-trivial operator V_{shift}, then the matrix L will in in general not be the identity, and the transformation law (A.31) violates gauge invariance.

Our method to recover gauge invariance is to replace the gauge potential A by a more general operator $\mathcal{A}$, which in momentum space has the form

$$\mathcal{A}\Big(p + \frac{q}{2}, p - \frac{q}{2}\Big) := \hat{A}\big(L(p)^{-1} q\big) , \tag{A.32}$$

where $\hat{A}$ is the Fourier transform of the classical potential to be used when no regularization is present. In the case $A = \partial\!\!\!/\Lambda$ of a pure gauge field and fixing the momentum scale, we then find that $\mathcal{A}$ coincides with the multiplication operator $A(\big((L(p)^*)^{-1} x\big))$, just compensating the factors $L(p)^*$ in (A.31). In view of the regularity assumptions (A.10), the matrix L scales in powers of the regularization length. Thus A is a *nonlocal* operator, but only *on the microscopic scale* ε. On the macroscopic scale, however, it coincides with the classical local potential. We also point out that the compatibility conditions worked out in Section A.3 under the assumption that $\mathcal{B}$ is a multiplication operator are valid just as well for the nonlocal potential (A.32), because after fixing the momentum scale, $\mathcal{A}$ reduces to a multiplication operator, so that our previous considerations again apply.

A.6. **The Regularized Light-Cone Expansion.** We are now in the position to perform the light-cone expansion. Our starting point is the operator product expansion (A.27). We choose $\hat{\mathcal{B}}$ to be the Fourier transform of a general multiplication or differential operator and introduce $\mathcal{B}$ in analogy to (A.32) as a non-local operator. After fixing the momentum scale, this operator reduces again to a multiplication or differential operator. Then the light-cone expansion can be performed exactly as described in [2] and [4, §2.5]. Finally, one can transform the obtained formulas with V_{shift} and V_{shear} (again using the rules (A.29) and (A.30)). Since the resulting line integrals do not depend on the momentum scale, the regularization only affects the factors $T^{(n)}$. The condition that $\mathcal{B}$ should be regularity compatible can be described by the parameters τ_i^{reg}. We thus obtain the formalism described in §2.5 and §2.6.

We finally compare our constructions with those in [4, Appendix D]. Clearly, the constructions here are much more general because they apply to any order in perturbation theory and may involve a chiral asymmetry. Moreover, the momentum shift operator V_{shift} makes it possible to describe general surface states, and also we allow for a large shear of the surface states (whereas in [4, Appendix D] we always assumed the shear to be small). Nevertheless, the basic idea that in order to preserve the gauge invariance in the presence of a regularization, one should replace the classical potentials by operators which are nonlocal on the microscopic scale already appeared in [4, Appendix D] (see the explanation after [4, eq. (D.26)]). Thus [4, Appendix D] can be regarded as a technical and conceptual preparation which is superseded by the constructions given here.

Notation Index

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